Analytical Mechanics: Variational Principles

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Agenda

1. Variational Principle in Statics
2. Variational Principle in Statics under Constraints
3. Variational Principle in Dynamics
4. Variational Principle in Dynamics under Constraints

Statics

Variation principle in statics

$$\text{minimize} \quad I = U - W$$

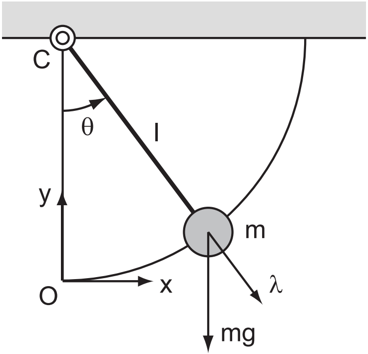
under constraint

$$\text{minimize} \quad I = U - W$$
$$\text{subject to} \quad R = 0$$

Solutions

- analytically solve $\delta I = 0$
- numerical optimization (`fminbnd` or `fmincon`)

Example (simple pendulum)



simple pendulum of length *l* and mass *m* suspended at point C
τ: external torque around C, θ: angle around C
Given τ, derive θ at equilibrium.

Statics in variational form

U	potential energy
W	work done by external forces/torques

Variational principle in statics

Internal energy $I = U - W$ reaches to its minimum at equilibrium:

$$I = U - W \rightarrow \text{minimum}$$

Statics in variational form

Solutions:

1. Solve

$$\text{minimize} \quad I = U - W$$

analytically

2. Solve

$$\text{minimize} \quad I = U - W$$

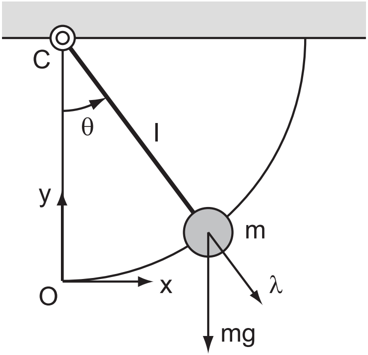
numerically

3. Solve

$$\delta I = 0$$

analytically

Example (simple pendulum)



$$U = mgl(1-\cos\theta), \quad W = \tau\theta$$

$$I = mgl(1-\cos\theta) - \tau\theta$$

Example (simple pendulum)

Solve

$$\text{minimize} \quad I = mgl(1 - \cos\theta) - \tau\theta$$

analytically

$$\Downarrow$$

$$\frac{\partial I}{\partial \theta} = mgl\sin\theta - \tau = 0$$

Equilibrium of moment around C

Example (simple pendulum)

Solve

$$\text{minimize} \quad I = mgl(1 - \cos\theta) - \tau\theta$$
$$(-\pi \leq \theta \leq \pi)$$

numerically

$$\Downarrow$$

Apply `fminbnd` to minimize a function numerically

Example (simple pendulum)

Sample Programs

- minimizing internal energy
- internal energy of simple pendulm

Example (simple pendulum)

Result

```
>> internal_energy_simple_pendulum_min

thetamin =

    0.5354


Imin =

   -0.0261
```

Example (simple pendulum)

Solve

$$\delta I = 0$$

analytically

$$\Downarrow$$

$$\begin{aligned} I &= mgl(1 - \cos\theta) - \tau\theta \\ I + \delta I &= mgl(1 - \cos(\theta + \delta\theta)) - \tau(\theta + \delta\theta) \end{aligned}$$

Example (simple pendulum)

Note that $\cos(\theta + \delta\theta) = \cos\theta - (\sin\theta)\delta\theta$:

$$I = mgl(1 - \cos\theta) - \tau\theta$$
$$I + \delta I = mgl(1 - \cos\theta + (\sin\theta)\delta\theta) - \tau(\theta + \delta\theta)$$

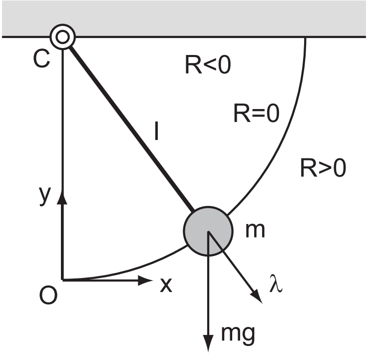
$$\Downarrow$$

$$\begin{aligned} \delta I &= mgl(\sin\theta)\delta\theta - \tau\delta\theta \\ &= (mgl\sin\theta - \tau)\delta\theta \equiv 0, \qquad \forall\delta\theta \end{aligned}$$

$$\Downarrow$$

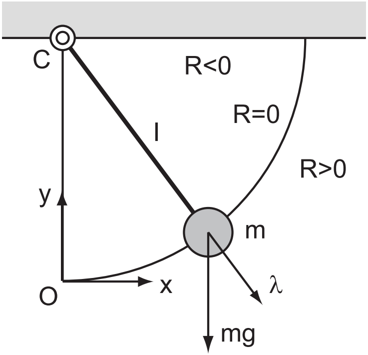
$$mgl\sin\theta - \tau = 0$$

Example (pendulum in Cartesian coordinates)



simple pendulum of length l and mass m suspended at point C
$[\,x,\ y\,]^{\mathrm{T}}$: position of mass
$[\,f_x,\ f_y\,]^{\mathrm{T}}$: external force applied to mass
Given $[\,f_x,\ f_y\,]^{\mathrm{T}}$, derive $[\,x,\ y\,]^{\mathrm{T}}$ at equilibrium.

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass $= l$

$$R \triangleq \left\{ x^2 + (y - l)^2 \right\}^{1/2} - l = 0$$

Statics under single constraint

U	potential energy
W	work done by external forces/torques
R	geometric constraint

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraint $R = 0$:

$$\begin{aligned} &\text{minimize} \quad U - W \\ &\text{subject to} \quad R = 0 \end{aligned}$$

Statics under single constraint

Solutions:

1. Solve

$$\begin{aligned} \text{minimize} \quad & U - W \\ \text{subject to} \quad & R = 0 \end{aligned}$$

analytically

2. Solve

$$\begin{aligned} \text{minimize} \quad & U - W \\ \text{subject to} \quad & R = 0 \end{aligned}$$

numerically

Statics under single constraint

Solve

$$\text{minimize} \quad U - W$$
$$\text{subject to} \quad R = 0$$

analytically

$$\Downarrow$$

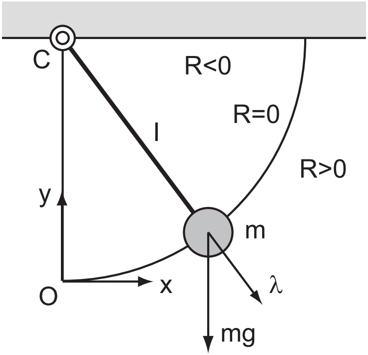
$$\text{minimize} \quad I = U - W - \lambda R$$

λ: Lagrange's multiplier

$$\Downarrow$$

$$\delta I = \delta(U - W - \lambda R) = 0$$

Example (pendulum in Cartesian coordinates)



$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \left\{x^2 + (y - l)^2\right\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

$$I = mgy - (f_x x + f_y y) - \lambda \left[\left\{ x^2 + (y-l)^2 \right\}^{1/2} - l \right]$$

Note that $\delta R = R_x \, \delta x + R_y \, \delta y$, where

$$\begin{aligned} R_x &\stackrel{\triangle}{=} \frac{\partial R}{\partial x} = x \left\{ x^2 + (y-l)^2 \right\}^{-1/2} \\ R_y &\stackrel{\triangle}{=} \frac{\partial R}{\partial y} = (y-l) \left\{ x^2 + (y-l)^2 \right\}^{-1/2} \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \delta I &= mg \, \delta y - f_x \delta x - f_y \delta y - \lambda R_x \delta x - \lambda R_y \delta y \\ &= (-f_x - \lambda R_x)\delta x + (mg - f_y - \lambda R_y)\delta y \equiv 0, \qquad \forall \delta x, \delta y \end{aligned}$$

Example (pendulum in Cartesian coordinates)

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \end{aligned}$$

$$\Downarrow$$

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} \text{grav. force,} \quad \begin{bmatrix} f_x \\ f_y \end{bmatrix} \text{ext. force,} \quad \lambda \underset{\text{gradient vector } (\perp \text{ to } R = 0)}{\underline{\begin{bmatrix} R_x \\ R_y \end{bmatrix}}} \text{constraint force}$$

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \\ R &= 0 \end{aligned}$$

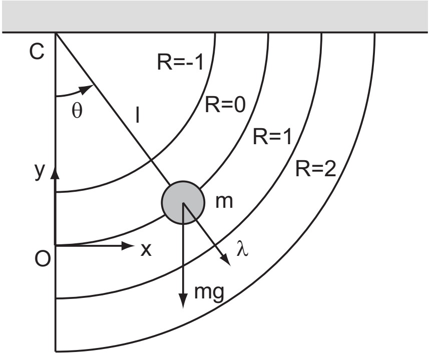
$$\Downarrow$$

we can determine position of mass $[\, x,\ y \,]^{\mathrm{T}}$ and magnitude of constraint force λ

Example (pendulum in Cartesian coordinates)

Note

$$\begin{aligned} I &= U - W - \lambda R \\ &= U - (W + \lambda R) \end{aligned}$$



λ	magnitude of a constraint force
R	distance along the force

constraint force $\perp$ contour $R =$ constant

λR	work done by a constraint force

$W + \lambda R$ work done by external & constraint forces

Statics under single constraint

Solve

$$\begin{aligned} \text{minimize} \quad & I = U - W \\ \text{subject to} \quad & R = 0 \end{aligned}$$

numerically

$$\Downarrow$$

Apply `fmincon` to minimize a function numerically under constraints
Note: "Optimization Toolbox" is needed to use `fmincon`

Example (pendulum in Cartesian coordinates)

Sample Programs

- minimizing internal energy (Cartesian)
- internal energy of simple pendulm (Cartesian)
- constraints

Example (pendulum in Cartesian coordinates)

Result:

```
>> internal_energy_pendulum_Cartesian_min
Local minimum found that satisfies the constraints.

<stopping criteria details>

qmin =
    1.4001
    3.4281

Imin =
   -0.4897
```

Statics under multiple constraints

U	potential energy
W	work done by external forces/torques
R_1, R_2	geometric constraints

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraints $R_1 = 0$ and $R_2 = 0$:

$$\text{minimize} \quad U - W$$
$$\text{subject to} \quad R_1 = 0, \quad R_2 = 0$$

$$\delta I = \delta(U - W - \lambda_1 R_1 - \lambda_2 R_2) = 0$$

λ_1, λ_2: Lagrange's multipliers

Dynamics

Lagrangian

$$\mathcal{L} = T - U + W$$

$$\mathcal{L} = T - U + W + \lambda R \qquad \text{(under constraint)}$$

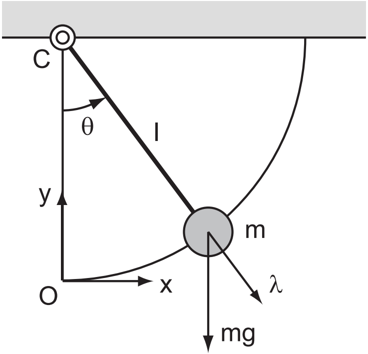
Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0}$$

Solutions

- numerical ODE solver (`ode45`)
- constraint stabilization method (CSM)

Example (simple pendulum)



simple pendulum of length *l* and mass *m* suspended at point C
τ: external torque around C at time *t*, θ: angle around C at time *t*
Derive the motion of the pendulum.

Dynamics in variational form

T	kinetic energy
U	potential energy
W	work done by external forces/torques

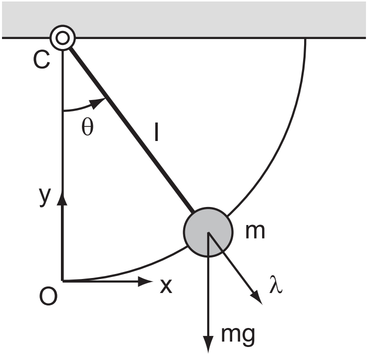
Lagrangian

$$\mathcal{L} = T - U + W$$

Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

Example (simple pendulum)



$$T = \frac{1}{2}(ml^2)\dot{\theta}^2$$

$$U = mgl(1 - \cos\theta), \quad W = \tau\theta$$

Example (simple pendulum)

Lagrangian

$$\mathcal{L} = \frac{1}{2}(ml^2)\dot{\theta}^2 - mgl(1 - \cos\theta) + \tau\theta$$

partial derivatives

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta + \tau, \quad \frac{\partial L}{\partial \dot{\theta}} = (ml^2)\dot{\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta}$$

Lagrange equation of motion

$$-mgl\sin\theta + \tau - ml^2\ddot{\theta} = 0$$

Example (simple pendulum)

Equation of the pendulum motion

$$ml^2\ddot{\theta} = -mgl\sin\theta + \tau$$

$$\Downarrow$$

Canonical form of ordinary differential equation

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{ml^2}\left(\tau - mgl\sin\theta\right)\end{aligned}$$

can be solved numerically by an ODE solver

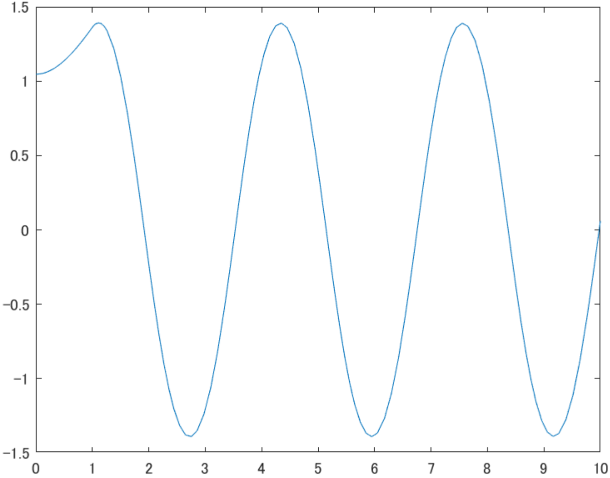
Example (simple pendulum)

Sample Programs

- solve the equation of motion of simple pendulum
- equation of motion of simple pendulum
- external torque

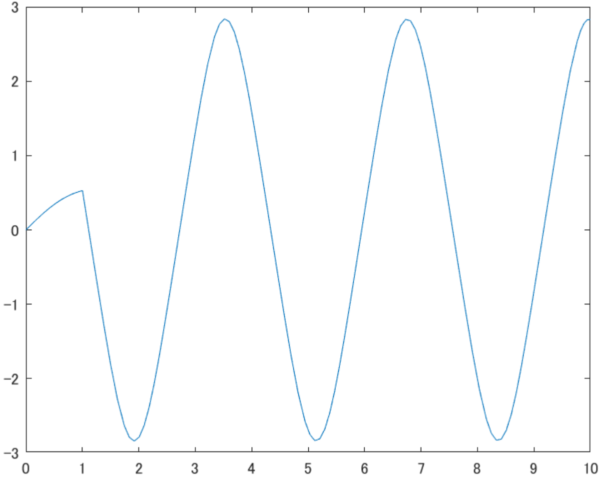
Example (simple pendulum)

Result



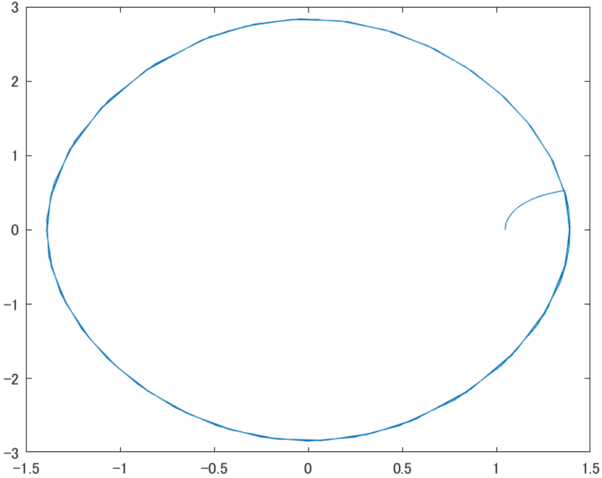
Example (simple pendulum)

Result



Example (simple pendulum)

Result



Example (pendulum with viscous friction)

Assumptions

viscous friction around supporting point C works
viscous friction causes a negative torque around C
magnitude of the torque is proportional to angular velocity

$$\text{viscous friction torque} = -b\dot{\theta} \quad (b\text{: positive constant})$$

Replacing τ by $\tau - b\dot{\theta}$:

$$(ml^2)\ddot{\theta} = (\tau - b\dot{\theta}) - mgl\sin\theta$$

$$\Downarrow$$

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{ml^2}\left(\tau - b\omega - mgl\sin\theta\right) \end{aligned}$$

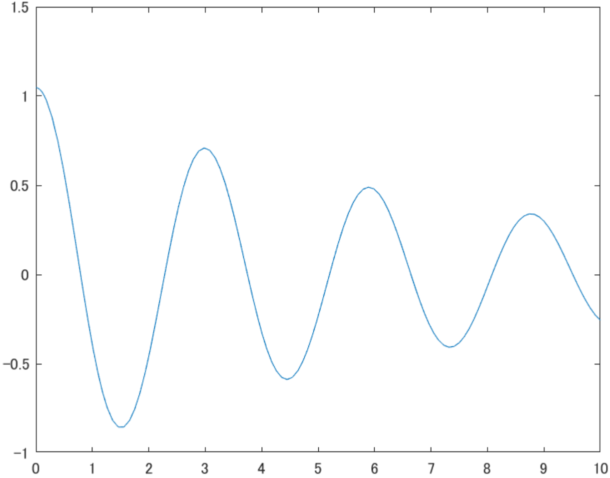
Example (pendulum with viscous friction)

Sample Programs

- solve the equation of motion of damped pendulum
- equation of motion of damped pendulum
- external torque

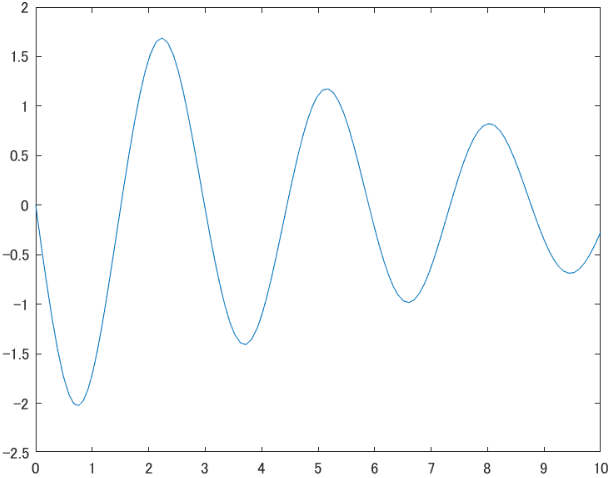
Example (pendulum with viscous friction)

Result



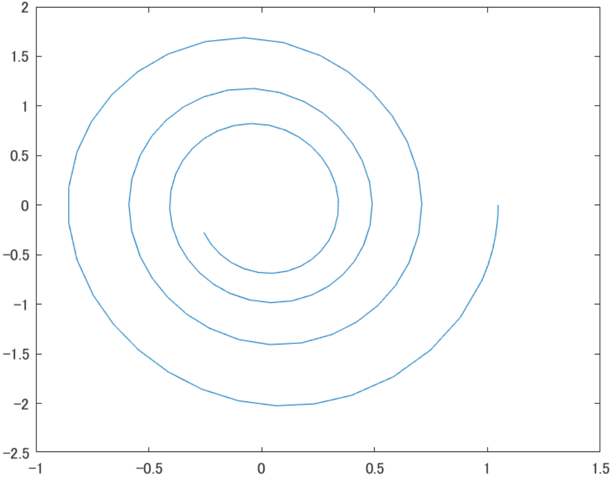
Example (pendulum with viscous friction)

Result

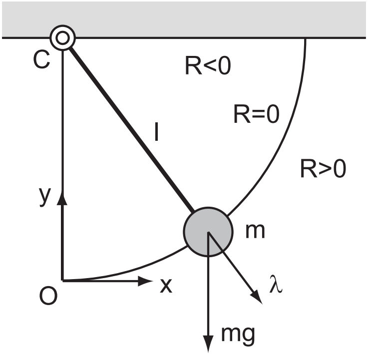


Example (pendulum with viscous friction)

Result

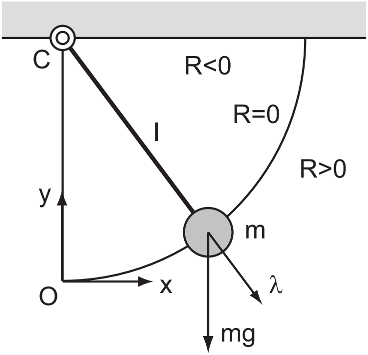


Example (pendulum in Cartesian coordinates)



simple pendulum of length l and mass m suspended at point C
$[\,x,\ y\,]^{\mathrm{T}}$: position of mass at time t
$[\,f_x,\ f_y\,]^{\mathrm{T}}$: external force applied to mass at time t
Derive the motion of the pendulum in Cartesian coordinates.

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass $= l$

$$R \triangleq \left\{ x^2 + (y - l)^2 \right\}^{1/2} - l = 0$$

Dynamics under single constraint

T	kinetic energy
U	potential energy
W	work done by external forces/torques

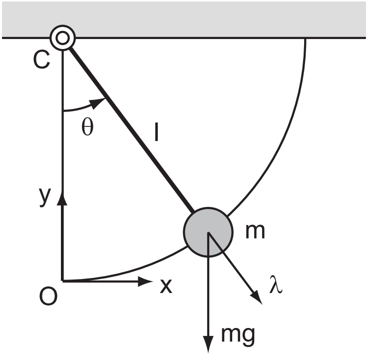
Lagrangian

$$\mathcal{L} = T - U + W + \lambda R$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0$$

Example (pendulum in Cartesian coordinates)



$$T = \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2\}$$

$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \left\{x^2 + (y-l)^2\right\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

Lagrangian

$$\mathcal{L} = \frac{1}{2} m \{ \dot{x}^2 + \dot{y}^2 \} - mgy + f_x x + f_y y + \lambda \left[\left\{ x^2 + (y - l)^2 \right\}^{1/2} - l \right]$$

partial derivatives

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= f_x + \lambda R_x, \quad & \frac{\partial \mathcal{L}}{\partial \dot{x}} &= m\dot{x} \\ \frac{\partial \mathcal{L}}{\partial y} &= -mg + f_y + \lambda R_y, \quad & \frac{\partial \mathcal{L}}{\partial \dot{y}} &= m\dot{y} \end{aligned}$$

Lagrange equations of motion

$$\begin{aligned} f_x + \lambda R_x - m\ddot{x} &= 0 \\ -mg + f_y + \lambda R_y - m\ddot{y} &= 0 \end{aligned}$$

Example (pendulum in Cartesian coordinates)

Lagrange equations of motion

$$\underset{\text{gravitational}}{\begin{bmatrix} 0 \\ -mg \end{bmatrix}} + \underset{\text{external}}{\begin{bmatrix} f_x \\ f_y \end{bmatrix}} + \underset{\text{constraint}}{\lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix}} + \underset{\text{inertial}}{\left\{ -m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} \right\}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

dynamic equilibrium among forces

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$\begin{aligned} m\ddot{x} &= f_x + \lambda R_x \\ m\ddot{y} &= -mg + f_y + \lambda R_y \\ R &= 0 \end{aligned}$$

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$\begin{aligned} m\ddot{x} &= f_x + \lambda R_x \\ m\ddot{y} &= -mg + f_y + \lambda R_y \\ R &= 0 \end{aligned}$$

Mixture of differential and algebraic equations

$$\Downarrow$$

Difficult to solve the mixture of differential and algebraic equations

Constraint stabilization method (CSM)

Constraint stabilization

convert algebraic eq. to its almost equivalent differential eq.

$$\text{algebraic eq.} \quad R = 0$$

$$\Downarrow$$

$$\text{differential eq.} \quad \ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0$$

(α: large positive constant)

critical damping (converges to zero most quickly)

Constraint stabilization method (CSM)

Dynamic equation of motion under geometric constraint:

$$\begin{array}{rl} \text{differential eq.} & \dfrac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0} \\ \text{algebraic eq.} & R = 0 \end{array}$$

$$\Downarrow$$

$$\begin{array}{rl} \text{differential eq.} & \dfrac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0} \\ \text{differential eq.} & \ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0 \end{array}$$

can be solved numerically by an ODE solver.

Computing equation for constraint stabilization

Assume R depends on x and y: $R(x, y) = 0$
Differentiating $R(x, y)$ w.r.t time t:

$$\dot{R} = \frac{\partial R}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_x\dot{x} + R_y\dot{y}$$

Differentiating $R_x(x, y)$ and $R_y(x, y)$ w.r.t time t:

$$\begin{aligned}\dot{R}_x &= \frac{\partial R_x}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R_x}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_{xx}\dot{x} + R_{xy}\dot{y} \\ \dot{R}_y &= \frac{\partial R_y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R_y}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_{yx}\dot{x} + R_{yy}\dot{y}\end{aligned}$$

Second order time derivative:

$$\begin{aligned}\ddot{R} &= (\dot{R}_x\dot{x} + R_x\ddot{x}) + (\dot{R}_y\dot{y} + R_y\ddot{y}) \\ &= (R_{xx}\dot{x} + R_{xy}\dot{y})\dot{x} + R_x\ddot{x} + (R_{yx}\dot{x} + R_{yy}\dot{y})\dot{y} + R_y\ddot{y}\end{aligned}$$

Computing equation for constraint stabilization

Second order time derivative:

$$\ddot{R} = \begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Equation to stabilize constraint:

$$-\begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + 2\alpha(R_x \dot{x} + R_y \dot{y}) + \alpha^2 R$$

$$\Downarrow \quad v_x \overset{\triangle}{=} \dot{x}, \; v_y \overset{\triangle}{=} \dot{y}$$

$$-\begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R$$

Example (pendulum in Cartesian coordinates)

Equation for stabilizing constraint $R(x, y) = 0$:

$$-R_x \dot{v}_x - R_y \dot{v}_y = C(x, y, v_x, v_y)$$

where

$$C(x, y, v_x, v_y) = \left[\begin{array}{cc} v_x & v_y \end{array} \right] \left[\begin{array}{cc} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{array} \right] \left[\begin{array}{c} v_x \\ v_y \end{array} \right] + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R$$

In this example

$$P = \{x^2 + (y - l)^2\}^{-1/2}, \quad R_x = xP, \quad R_y = (y - l)P$$
$$R_{xx} = P - x^2 P^3, \quad R_{yy} = P - (y - l)^2 P^3$$
$$R_{xy} = R_{yx} = -x(y - 1)P^3$$

Example (pendulum in Cartesian coordinates)

Combining equations of motion and equation for constraint stabilization:

$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\begin{bmatrix} m & & -R_x \\ & m & -R_y \\ -R_x & -R_y & \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \lambda \end{bmatrix} = \begin{bmatrix} f_x \\ -mg + f_y \\ C(x, y, v_x, v_y) \end{bmatrix}$$

five equations w.r.t. five unknown variables x, y, v_x, v_y and λ

given x, y, v_x, $v_y \Longrightarrow \dot{x}$, $\dot{y}$, $\dot{v}_x$, $\dot{v}_y$

This canonical ODE can be solved numerically by an ODE solver.

Example (pendulum in Cartesian coordinates)

Let $\boldsymbol{x} = [\,x,\ y\,]^{\mathrm{T}}$. Introducing gradient vector

$$\boldsymbol{g} = \begin{bmatrix} R_x \\ R_y \end{bmatrix}$$

yields

$$\dot{R} = \boldsymbol{g}^{\mathrm{T}} \dot{\boldsymbol{x}}$$

Introducing Hessian matrix

$$H = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}$$

yields

$$\ddot{R} = \boldsymbol{g}^{\mathrm{T}} \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^{\mathrm{T}} H \dot{\boldsymbol{x}}$$

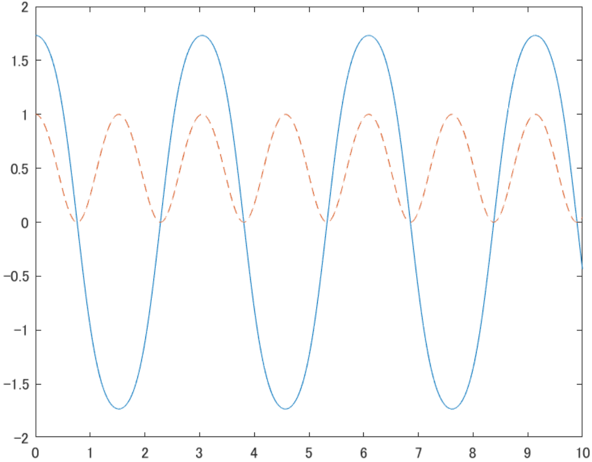
Example (pendulum in Cartesian coordinates)

Sample Programs

- solve the equation of motion of simple pendulum (Cartesian)
- equation of motion of simple pendulum (Cartesian)

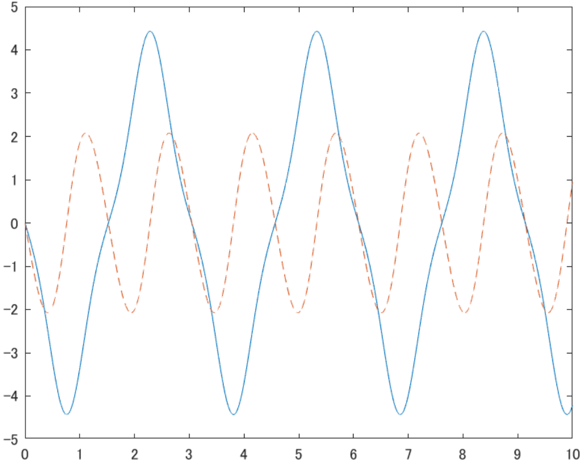
Example (pendulum in Cartesian coordinates)

t–x, y



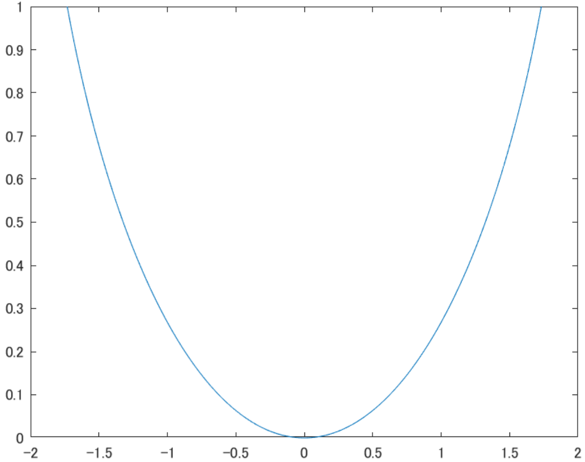
Example (pendulum in Cartesian coordinates)

t–v_x, v_y



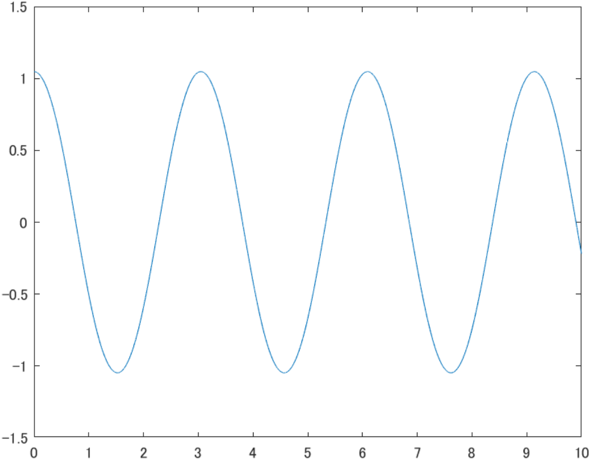
Example (pendulum in Cartesian coordinates)

x–y



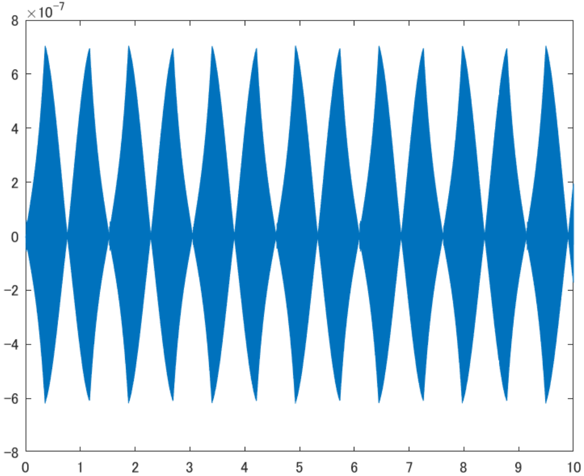
Example (pendulum in Cartesian coordinates)

t–computed θ



Example (pendulum in Cartesian coordinates)

t–constraint R



Notice

Lagrangian

$$
\begin{aligned}
\mathcal{L} &= T - U + W + \lambda R \\
&= T - (U - W - \lambda R) \\
&= T - I
\end{aligned}
$$

Lagrangian is equal to the difference between kinetic energy and internal energy under a constraint

Summary

Variational principles

- statics $I = U - W$
- statics under constraint $I = U - W - \lambda R$

$$\delta I \quad \equiv \quad 0$$

- dynamics $\mathcal{L} = T - U + W$
- dynamics under constraint $\mathcal{L} = T - U + W + \lambda R$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0}$$

- constraint stabilization method

Summary

How to solve a static problem

Solve (nonlinear) equations originated from variation

or

Numerically minimize internal energy

How to solve a dynamic problem

Step 1 Derive Lagrange equations of motion analytically

Step 2 Solve the derived equations numerically

Report

Report # 1 due date : Oct. 24 (Mon) 1:00 AM

Simulate the dynamic motion of a pendulum under viscous friction described with Cartesian coordinates x and y. Apply constraint stabilization method to convert the constraint into its almost equivalent ODE, then apply any ODE solver to solve a set of ODEs (equations of motion and equation for constraint stabilization) numerically.

Submit your report in pdf format to manaba+R

File name shoud be:
student number (11 digits) your name (without space).pdf
For example 12345678901HiraiShinichi.pdf

Report

Report # 2 due date : Oct. 31 (Mon) 1:00 AM

Assume that a system is described by four coordinates q_1 through q_4. Two constraints R_1 and R_2 are imposed on the system. Let $\boldsymbol{q} = [\, q_1,\ q_2,\ q_3,\ q_4\,]^{\mathrm{T}}$ and $\boldsymbol{R} = [\, R_1,\ R_2\,]^{\mathrm{T}}$. Let $\boldsymbol{g}_1$ and H_1 be gradient vector and Hessian matrix related to R_1 while $\boldsymbol{g}_2$ and H_2 be gradient vector and Hessian matrix related to R_2. Let J be Jacobian given by

$$J = \begin{bmatrix} \partial R_1/\partial q_1 & \partial R_1/\partial q_2 & \partial R_1/\partial q_3 & \partial R_1/\partial q_4 \\ \partial R_2/\partial q_1 & \partial R_2/\partial q_2 & \partial R_2/\partial q_3 & \partial R_2/\partial q_4 \end{bmatrix}$$

Show the following equations:

$$\dot{\boldsymbol{R}} = J\dot{\boldsymbol{q}}$$

$$\ddot{\boldsymbol{R}} = J\ddot{\boldsymbol{q}} + \begin{bmatrix} \dot{\boldsymbol{q}}^{\mathrm{T}} H_1 \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^{\mathrm{T}} H_2 \dot{\boldsymbol{q}} \end{bmatrix}$$

Appendix: Variational calculus

Small virtual deviation of variables or functions.

$$y = x^2$$

Let us change variable x to $x + \delta x$, then variable y changes to $y + \delta y$ accordingly.

$$\begin{aligned} y + \delta y &= (x + \delta x)^2 \\ &= x^2 + 2x\,\delta x + (\delta x)^2 \\ &= x^2 + 2x\,\delta x \end{aligned}$$

Thus

$$\delta y = 2x\,\delta x$$

Appendix: Variational calculus

Small virtual deviation of variables or functions.

$$I = \int_0^T \{x(t)\}^2 \, \mathrm{d}t$$

Let us change function $x(t)$ to $x(t) + \delta x(t)$, then variable I changes to $I + \delta I$ accordingly.

$$\begin{aligned} I + \delta I &= \int_0^T \{x(t) + \delta x(t)\}^2 \, \mathrm{d}t \\ &= \int_0^T \{x(t)\}^2 + 2x(t)\,\delta x(t)\,\mathrm{d}t \end{aligned}$$

Thus

$$\delta I = \int_0^T 2x(t)\,\delta x(t)\,\mathrm{d}t$$

Appendix: Variational calculus

Variational operator δ

$\delta\theta$	virtual deviation of variable θ
$\delta f(\theta)$	virtual deviation of function $f(\theta)$

$$\delta f(\theta) = f'(\theta)\delta\theta$$

virtual increment of variable $\theta \to \theta + \delta\theta$

increment of function $f(\theta) \to f(\theta + \delta\theta) = f(\theta) + f'(\theta)\delta\theta$

$$f(\theta) \to f(\theta) + \delta f(\theta)$$

simple examples

$$\delta(5x) = 5\,\delta x \qquad \delta x^2 = 2x\,\delta x$$

$$\delta\sin\theta = (\cos\theta)\,\delta\theta, \qquad \delta\cos\theta = (-\sin\theta)\,\delta\theta$$

Appendix: Variational calculus

Variational operator δ

$\delta\theta$	virtual deviation of variable θ
$\delta f(\theta)$	virtual deviation of function $f(\theta)$

$$\delta f(\theta) = f'(\theta)\delta\theta$$

virtual increment of variable $\theta \to \theta + \delta\theta$

increment of function
$$f(\theta) \to f(\theta + \delta\theta) = f(\theta) + f'(\theta)\delta\theta$$
$$f(\theta) \to f(\theta) + \delta f(\theta)$$

simple examples

$$\delta(5x) = 5\,\delta x \qquad \delta x^2 = 2x\,\delta x$$
$$\delta\sin\theta = (\cos\theta)\,\delta\theta, \qquad \delta\cos\theta = (-\sin\theta)\,\delta\theta$$

Appendix: Variational calculus

assume that θ depends on time t
virtual increment of function $\theta(t) \to \theta(t) + \delta\theta(t)$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \to \frac{\mathrm{d}}{\mathrm{d}t}(\theta + \delta\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\delta\theta$$

$$\int \theta \, \mathrm{d}t \to \int (\theta + \delta\theta) \, \mathrm{d}t = \int \theta \, \mathrm{d}t + \int \delta\theta \, \mathrm{d}t$$

variation of derivative and integral

$$\delta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\delta\theta$$

$$\delta \int \theta \, \mathrm{d}t = \int \delta\theta \, \mathrm{d}t$$

variational operator and differential/integral operator can commute

Appendix: Lagrange multiplier method

converts minimization (maximization) under conditions into minimization (maximization) without conditions.

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) \\ &\text{subject to} \quad g(\boldsymbol{x}) = 0 \end{aligned}$$

$$\Downarrow$$

$$\text{minimize} \quad I(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \lambda g(\boldsymbol{x})$$

$$\Downarrow$$

$$\begin{aligned} \frac{\partial I}{\partial \boldsymbol{x}} &= \frac{\partial f}{\partial \boldsymbol{x}} + \lambda \frac{\partial g}{\partial \boldsymbol{x}} = \boldsymbol{0} \\ \frac{\partial I}{\partial \lambda} &= g(\boldsymbol{x}) = 0 \end{aligned}$$

Appendix: Lagrange multiplier method (example)

Length of each edge of a cube is given by x, y, and z.
Determine x, y, and z that minimizes the surface of the cube when the cube volume is constantly specified by a^3:

$$\begin{aligned} \text{minimize} \quad & S(x,y,z) = 2xy + 2yz + 2zx \\ \text{subject to} \quad & R(x,y,z) \overset{\triangle}{=} xyz - a^3 = 0 \end{aligned}$$

Introducing Lagrange multiplier λ, the above conditional minimization can be converted into the following unconditional minimization:

$$\begin{aligned} \text{minimize} \quad I(x,y,z,\lambda) &= S(x,y,z) + \lambda R(x,y,z) \\ &= 2xy + 2yz + 2zx + \lambda(xyz - a^3) \end{aligned}$$

Appendix: Lagrange multiplier method (example)

Calculating partial derivatives:

$$\frac{\partial I}{\partial x} = 2y + 2z - \lambda yz = 0 \tag{1}$$

$$\frac{\partial I}{\partial y} = 2z + 2x - \lambda zx = 0 \tag{2}$$

$$\frac{\partial I}{\partial z} = 2x + 2y - \lambda xy = 0 \tag{3}$$

$$\frac{\partial I}{\partial \lambda} = xyz - a^3 = 0 \tag{4}$$

Calculating $(1) \cdot x - (2) \cdot y$, we have

$$z(x - y) = 0,$$

which directly yields $x = y$. Similarly, we have $y = z$ and $z = x$. Consequently, we concludes $x = y = z = a$.

Appendix: ODE solver

Let us solve van del Pol equation:

$$\ddot{x} - 2(1 - x^2)\dot{x} + x = 0$$

Canonical form:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= 2(1 - x^2)\dot{x} - x \end{aligned}$$

State variable vector:

$$\boldsymbol{q} = \left[\begin{array}{c} x \\ v \end{array} \right]$$

Appendix: ODE solver (MATLAB)

File `van_der_Pol.m` describes the canonical form:

```
function dotq = van_der_Pol (t,q)
    x = q(1);
    v = q(2);
    dotx = v;
    dotv = 2*(1-x^2)*v - x;
    dotq = [dotx; dotv];
end
```

File name `van_der_Pol` should conincide with function name `van_der_Pol`.

Appendix: ODE solver (MATLAB)

File `van_der_Pol_solve.m` solves van der Pol equation numerically:

```
timestep=0.00:0.10:10.00;
qinit=[2.00;0.00];
[time,q]=ode45(@van_der_Pol,timestep,qinit);

% line style   solid -   broken -.   chain --   dotted :
plot(time,q(:,1),'-', time,q(:,2),'-.');
```

Appendix: ODE solver (MATLAB)

```
>> time
time =
         0
    0.1000
    0.2000
    0.3000
    0.4000

>> q
q =
    2.0000         0
    1.9917   -0.1504
    1.9721   -0.2338
    1.9461   -0.2822
    1.9163   -0.3125
```

The first and second columns corresponds to x and v.

Appendix: ODE solver (MATLAB)