## Chapter 1

## Motion and Deformation

### 1.1 Dimension of soft robot bodies

Soft robots will move and deform. Let us formulate the motion and deformation of soft robots. All actual soft robots have three-dimensional bodies acting in three-dimensional space. It is natural to build three-dimensional models of soft robots. Yet when main motion and deformation of a soft robot is one-dimensional or two-dimensional, we build a one-dimensional or two-dimensional model to focus on the main motion and deformation. Consequently, we have the following models: one-dimensional model (Fig. 1.1(a)), two-dimensional model (Fig. 1.1(b)), three-dimensional model (Fig. 1.1(c)). One-dimensional models focus on translational motion and extensional deformation in one-dimensional space. Two-dimensional models focus on translational and rotational motion as well as extensional and shear deformation in two-dimensional space. Three-dimensional models focus on translational and rotational motion as well as extensional and shear deformation in three-dimensional space. Note that rotational motion and shear deformation appear in two- or three-dimensional models. Soft robots consisting of soft materials are referred to as soft-material robots. These three models are applicable to soft-material robots.

When one- or two-dimensions of bodies are dominant than others, the bodies may deform. Soft robots with such bodies are referred to as geometrically deformable robots. We build models focusing on dominant dimensions. Deformable linear models have one dominant dimension, that is, the other two dimensions are negligible. Deformable linear models in twodimensional space (Fig. 1.1(d)) focus on bending and extensional deformations. Deformable linear models in three-dimensional space (Fig. 1.1(e)) focus on bending, twisting, and extensional deformations. Deformable planar models (Fig. 1.1(f)) have two dominant dimensions, that is, the another dimension is negligible. Deformable planar models focus on bending and extensional deformations in three-dimensional space.

Table 1.1 summarizes relationship between dominant dimension of soft robot bodies and dimension of space. Elements $(1,1),(2,2)$, and $(3,3)$ imply one-, two-, and three-dimensional soft-material robots. Elements $(1,2)$ and $(1,3)$ correspond to deformable linear robots acting in two- and three-dimensional space. Element $(2,3)$ corresponds to deformable planar robots acting in three-dimensional space.

### 1.2 One-dimensional soft robot model

First, we will investigate the motion and deformation of a one-dimensional soft robot on a one-dimensional space. A one-dimensional soft robot is given by a line segment AB shown


Figure 1.1: Soft robot models

Table 1.1: Dimensions of soft robot bodies and space: $(1,1),(2,2),(3,3)$ imply 1D, 2D, 3 D soft-material robots. $(1,2),(1,3)$ correspond to deformable linear robots while $(2,3)$ corresponds to deformable planar robots.

in Fig. 1.2(a). Assume this one-dimensional soft robot moves and deforms as shown in Fig. 1.2(b). Let $u_{\mathrm{A}}$ and $u_{\mathrm{B}}$ be displacements of point A and B. Assume that $u_{\mathrm{A}}=u_{\mathrm{B}}$ is satisfied as shown in Fig. 1.2(c). Can we conclude that this soft robot AB moves but does not deform?


Figure 1.2: One-dimensional soft robot
Even though $u_{\mathrm{A}}=u_{\mathrm{B}}$, this soft robot may deform. Let C be the center point of line segment AB as shown in Fig. 1.3(a). Figure 1.3(b) shows the deformation where left half AC expands while right half CB shrinks. In this deformation, displacement of point C is bigger than displacements of A and B , that is, $u_{\mathrm{C}}>u_{\mathrm{A}}=u_{\mathrm{B}}$, as shown in Fig. 1.3(d). Figure $1.3(\mathrm{c})$ shows the deformation where left half AC shrinks while right half CB expands. In this deformation, displacement of point C is smaller than displacements of A and B , that is, $u_{\mathrm{C}}<u_{\mathrm{A}}=u_{\mathrm{B}}$, as shown in Fig. 1.3(e). Point C can be an arbitrary point between A and B. So, this investigation suggests us that it is necessary to specify displacements of all points on soft robot AB to describe its motion and deformation.

Let us describe displacements of all points on soft robot AB . Let $L$ be the natural length of the soft robot. Let $x$ be the distance from point A in the natural shape. Let $\mathrm{P}(x)$ be point on the soft robot at distance $x$, as illustrated Fig. 1.4(a). Then, point A is described as $\mathrm{P}(0)$ while point B is described as $\mathrm{P}(L)$. Let $u(x)$ be the displacement of point $\mathrm{P}(x)$, as shown in Fig. 1.4(b). Then, the motion and deformation of this soft robot can be specified by function $u(x)$, where $x \in[0, L]$, as shown in Fig. 1.4(c).

Let $\mathrm{P}(x)$ and $\mathrm{P}(x+h)$ be two neighboring points on a one-dimensional soft robot, where $h$ is the infinitesimal distance between the two points in its natural state. Since displacements

(a) center point C

(b) left half expands

(d) displacements

(e) displacements

Figure 1.3: Deformation of one-dimensional soft robot


Figure 1.4: Displacement function of one-dimensional soft robot
of $\mathrm{P}(x)$ and $\mathrm{P}(x+h)$ are $u(x)$ and $u(x+h)$, distance between the two points in its moved and deformed state is given by $u(x+h)-u(x)$. Thus, extensional strain at point $\mathrm{P}(x)$ is described as

$$
\begin{equation*}
\varepsilon=\frac{u(x+h)-u(x)}{h}=\frac{\mathrm{d} u}{\mathrm{~d} x} \tag{1.2.1}
\end{equation*}
$$

Note that extensional strain $\mathrm{d} u / \mathrm{d} x$ is positive/negative when the soft robot extends/shrinks around point $\mathrm{P}(x)$. Consequently, displacement function $u(x)$ is increasing/decreasing at $x$ when the soft robot extends/shrinks around point $\mathrm{P}(x)$.

### 1.3 Two-dimensional soft robot model

Let us investigate the motion and deformation of a two-dimensional soft robot on a twodimensional space. A two-dimensional soft robot is given by a two-dimensional region S at its natural state (Fig. 1.5(a)). Let $\mathrm{P}(x, y)$ be an arbitrary point in the soft robot, where $(x, y) \in$ S. Assume this two-dimensional soft robot moves and deforms (Fig. 1.5(b)). Displacement of point $\mathrm{P}(x, y)$ is then given by a two-dimensional vector:

$$
\boldsymbol{u}(x, y)=\left[\begin{array}{l}
u(x, y)  \tag{1.3.1}\\
v(x, y)
\end{array}\right]
$$

The motion and deformation of a two-dimensional soft robot can be specified by a vector function $\boldsymbol{u}(x, y)$, that is, by its two components $u(x, y)$ and $v(x, y)$.

Two-dimensional soft robots perform translational motion, rotational motion, and deformation. Let us describe the deformation of a two-dimensional soft robot using partial derivatives $u_{x}=\partial u / \partial x, u_{y}=\partial u / \partial y, v_{x}=\partial v / \partial x$, and $v_{y}=\partial v / \partial y$. Translational motion of a two-dimensional soft robot does not affect these partial derivatives since any translational motion yields constant displacement components over region S. In other words, these partial derivatives are independent of translational motion while depend on rotational motion and deformation.

Assume that an infinitesimal square of its length $\delta$ (Fig. 1.6(a)) at point $\mathrm{P}(x, y)$ deforms and rotates, resulting a parallelogram (Fig. 1.6(b)). Relative displacements of Q and R with respect to P are given by

$$
\begin{aligned}
\boldsymbol{u}(x+\delta, y)-\boldsymbol{u}(x, y) & =\frac{\partial \boldsymbol{u}}{\partial x}(x, y) \delta \\
\boldsymbol{u}(x, y+\delta)-\boldsymbol{u}(x, y) & =\frac{\partial \boldsymbol{u}}{\partial y}(x, y) \delta
\end{aligned}
$$



Figure 1.5: Displacement function of two-dimensional soft robot


Figure 1.6: Deformation and rotation of small square region


Figure 1.7: Extensions, shear deformation, and rotation

Thus, normalized displacements of Q and R with respect to P are described as follows:

$$
\begin{align*}
& \frac{\boldsymbol{u}(x+\delta, y)-\boldsymbol{u}(x, y)}{\delta}=\left[\begin{array}{l}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{array}\right]  \tag{1.3.2}\\
& \frac{\boldsymbol{u}(x, y+\delta)-\boldsymbol{u}(x, y)}{\delta}=\left[\begin{array}{l}
\frac{\partial v}{\partial x} \\
\frac{\partial v}{\partial y}
\end{array}\right] \tag{1.3.3}
\end{align*}
$$

Deformation of a two-dimensional soft robot is classified into extension along $x$-axis (Fig. 1.7(a)), extension along $y$-axis (Fig. 1.7(b)), and shear deformation (Fig. 1.7(c)). Additionally, rotational motion (Fig. 1.7(d)) affects the partial derivatives. Comparing the above equations and Fig. 1.7, we have

$$
\begin{array}{ll}
\frac{\partial u}{\partial x}=\text { extension along } x \text {-axis, } & \frac{\partial v}{\partial y}=\text { extension along } y \text {-axis } \\
\frac{\partial v}{\partial x}=\text { shear }+ \text { rotation, } & \frac{\partial u}{\partial y}=\text { shear }- \text { rotation }
\end{array}
$$

Let $\varepsilon_{x x}$ and $\varepsilon_{y y}$ be normal strain components along $x$ - and $y$-axes at point P , and $\varepsilon_{x y}$ be shear strain at point P , we have

$$
\begin{equation*}
\varepsilon_{x x}=\frac{\partial u}{\partial x}, \quad \varepsilon_{y y}=\frac{\partial v}{\partial y}, \quad 2 \varepsilon_{x y}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{1.3.4}
\end{equation*}
$$

which are referred to as Cauchy strain components, or simply strain components. Let us define

$$
\varepsilon \triangleq\left[\begin{array}{c}
\varepsilon_{x x}  \tag{1.3.5}\\
\varepsilon_{y y} \\
2 \varepsilon_{x y}
\end{array}\right]
$$

which is referred to as a pseudo strain vector or simply strain vector. This strain vector describes the deformation of a soft robot at an arbitrary point P .

### 1.4 Three-dimensional soft robot model

Let us investigate the motion and deformation of a three-dimensional soft robot in a threedimensional space. A three-dimensional soft robot is given by a three-dimensional region V at its natural state (Fig. 1.8(a)). Let $\mathrm{P}(x, y, z)$ be an arbitrary point in the soft robot, where $(x, y, z) \in \mathrm{V}$. Assume this three-dimensional soft robot moves and deforms (Fig. 1.8(b)). Displacement of point $\mathrm{P}(x, y, z)$ is then given by a three-dimensional vector:

$$
\boldsymbol{u}(x, y, z)=\left[\begin{array}{c}
u(x, y, z)  \tag{1.4.1}\\
v(x, y, z) \\
w(x, y, z)
\end{array}\right]
$$

The motion and deformation of a three-dimensional soft robot can be specified by a vector function $\boldsymbol{u}(x, y, z)$, that is, by its three components $u(x, y, z), v(x, y, z)$, and $w(x, y, z)$.


Figure 1.8: Displacement function of three-dimensional soft robot

(a) natural

(b) deformed and rotated

Figure 1.9: Deformation and rotation of small cube region

Assume that an infinitesimal cube of its length $\delta$ (Fig. 1.9(a)) at point $\mathrm{P}(x, y, z)$ deforms and rotates, resulting a parallelepiped (Fig. 1.9(b)). Then, we derive the following equations:

$$
\begin{aligned}
& \frac{\partial u}{\partial x}=\text { extension along } x \text {-axis } \\
& \frac{\partial v}{\partial y}=\text { extension along } y \text {-axis } \\
& \frac{\partial w}{\partial z}=\text { extension along } z \text {-axis } \\
& \frac{\partial w}{\partial y}=\text { shear in } y z \text {-plane }+ \text { rotation in } y z \text {-plane } \\
& \frac{\partial v}{\partial z}=\text { shear in } y z \text {-plane }- \text { rotation in } y z \text {-plane } \\
& \frac{\partial u}{\partial z}=\text { shear in } z x \text {-plane }+ \text { rotation in } z x \text {-plane } \\
& \frac{\partial w}{\partial x}=\text { shear in } z x \text {-plane }- \text { rotation in } z x \text {-plane } \\
& \frac{\partial v}{\partial x}=\text { shear in } x y \text {-plane }+ \text { rotation in } x y \text {-plane } \\
& \frac{\partial u}{\partial y}=\text { shear in } x y \text {-plane }- \text { rotation in } x y \text {-plane }
\end{aligned}
$$

Let $\varepsilon_{x x}, \varepsilon_{y y}$, and $\varepsilon_{z z}$ be normal strain components along $x$-, $y$-, and $z$-axes at point P whereas $\varepsilon_{y z}, \varepsilon_{z x}$, and $\varepsilon_{x y}$, be shear strain components over $y z-, z x-$, and $x y$-planes at point P , we
have

$$
\begin{align*}
& \varepsilon_{x x}=\frac{\partial u}{\partial x}, \quad \quad \varepsilon_{y y}=\frac{\partial v}{\partial y}, \quad \quad \varepsilon_{z z}=\frac{\partial w}{\partial z}, \\
& 2 \varepsilon_{y z}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}, \quad 2 \varepsilon_{z x}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}, \quad 2 \varepsilon_{x y}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{1.4.2}
\end{align*}
$$

which are referred to as Cauchy strain components, or simply strain components. Let us define the following strain vector:

$$
\varepsilon=\left[\begin{array}{c}
\varepsilon_{x x}  \tag{1.4.3}\\
\varepsilon_{y y} \\
\varepsilon_{z z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{z x} \\
2 \varepsilon_{x y}
\end{array}\right]
$$

The upper three components correspond to normal deformation while the lower three denote shear deformation.

### 1.5 Strain potential energy

Soft robot deformation yields strain potential energy. We formulate strain potential energy of soft robots based on strain potential energy density.

Let us formulate the strain potential energy of a one-dimensional soft robot. Let $E$ be Young's modulus of the robot material. Recalling that extensional strain is given by $\varepsilon$, we find that the strain energy density at point $\mathrm{P}(x)$ is formulated as follows:

$$
\begin{equation*}
\frac{1}{2} E \varepsilon^{2} \tag{1.5.1}
\end{equation*}
$$

The above quantity has the dimension of energy/volume $=\mathrm{N} / \mathrm{m}^{2}$. Let $A$ be the crosssectional area of the robot at point $\mathrm{P}(x)$. Volume element at point $\mathrm{P}(x)$ is then described as

$$
\begin{equation*}
A \mathrm{~d} x \tag{1.5.2}
\end{equation*}
$$

Integrating the product of the energy density and the volume element over one-dimensional region $[0, L]$ yields strain potential energy. Consequently, strain potential energy stored in this one-dimensional soft robot is formulated as follows:

$$
\begin{equation*}
U=\int_{0}^{L} \frac{1}{2} E \varepsilon^{2} A \mathrm{~d} x=\int_{0}^{L} \frac{1}{2} E A\left(\frac{\partial u}{\partial x}\right)^{2} \mathrm{~d} x \tag{1.5.3}
\end{equation*}
$$

Note that $E$ and $A$ may depend on $x$.
Let us formulate the strain potential energy of a two-dimensional soft robot. Recall that the deformation of a robot body is specified by three-dimensional vector $\varepsilon=\left[\varepsilon_{x x}, \varepsilon_{y y}\right.$, $\left.2 \varepsilon_{x y}\right]^{\top}$. Assuming that the robot material shows linear isotropic elasticity (see Section 7.1 for details), the strain energy density at point $\mathrm{P}(x, y)$ is formulated as follows:

$$
\begin{equation*}
\frac{1}{2} \varepsilon^{\top}\left(\lambda I_{\lambda}+\mu I_{\mu}\right) \varepsilon \tag{1.5.4}
\end{equation*}
$$

where $\lambda$ and $\mu$ are Lamé's constants and

$$
I_{\lambda}=\left[\begin{array}{ll}
1 & 1  \tag{1.5.5}\\
1 & 1
\end{array}\right], \quad I_{\mu}=\left[\begin{array}{lll}
2 & & \\
& 2 & \\
& & 1
\end{array}\right]
$$

Lamé's constants $\lambda$ and $\mu$ are specific to robot material, and are related to Young's modulus $E$ and Poisson's ratio $\nu$ by

$$
\begin{equation*}
\lambda=\frac{\nu E}{(1+\nu)(1-2 \nu)}, \quad \mu=\frac{E}{2(1+\nu)} \tag{1.5.6}
\end{equation*}
$$

Note that tensile test provides the values of Young's modulus $E$ and Poisson's ratio $\nu$, directly yielding the values of Lamé's constants $\lambda$ and $\mu$ by the above equations.

Let $h$ be the thickness of the robot at point $\mathrm{P}(x, y)$. Volume element at point $\mathrm{P}(x, y)$ is then described as

$$
\begin{equation*}
h \mathrm{~d} S=h \mathrm{~d} x \mathrm{~d} y \tag{1.5.7}
\end{equation*}
$$

Integrating the product of the energy density and the volume element over two-dimensional region $S$ yields strain potential energy. Consequently, strain potential energy stored in this two-dimensional soft robot is formulated as follows:

$$
\begin{equation*}
U=\int_{S} \frac{1}{2} \varepsilon^{\top}\left(\lambda I_{\lambda}+\mu I_{\mu}\right) \varepsilon h \mathrm{~d} S \tag{1.5.8}
\end{equation*}
$$

Note that $\lambda, \mu$, and $h$ may depend on $(x, y)$.
Let us formulate the strain potential energy of a three-dimensional soft robot. Recall that the deformation of a robot body is specified by six-dimensional vector $\varepsilon=\left[\varepsilon_{x x}, \varepsilon_{y y}\right.$, $\left.\varepsilon_{z z}, 2 \varepsilon_{y z}, 2 \varepsilon_{z x}, 2 \varepsilon_{x y}\right]^{\top}$. Assuming that the robot material shows linear isotropic elasticity (see Section 7.1 for details), the strain energy density at point $\mathrm{P}(x, y, z)$ is formulated as follows:

$$
\begin{equation*}
\frac{1}{2} \varepsilon^{\top}\left(\lambda I_{\lambda}+\mu I_{\mu}\right) \varepsilon \tag{1.5.9}
\end{equation*}
$$

where $\lambda$ and $\mu$ are Lamé's constants and

$$
I_{\lambda}=\left[\begin{array}{lll|l}
1 & 1 & 1  \tag{1.5.10}\\
1 & 1 & 1 & \\
1 & 1 & 1 & \\
\hline & & &
\end{array}\right], \quad I_{\mu}=\left[\begin{array}{lll|ll}
2 & & & & \\
& 2 & & & \\
& & 2 & & \\
\hline & & & 1 & \\
\hline & & & 1 & \\
& & & & 1
\end{array}\right]
$$

Volume element at point $\mathrm{P}(x, y, z)$ is described as

$$
\begin{equation*}
\mathrm{d} V=\mathrm{d} x \mathrm{~d} y \mathrm{~d} z \tag{1.5.11}
\end{equation*}
$$

Integrating the product of the energy density and the volume element over three-dimensional region $V$ yields strain potential energy. Consequently, strain potential energy stored in this three-dimensional soft robot is formulated as follows:

$$
\begin{equation*}
U=\int_{V} \frac{1}{2} \varepsilon^{\top}\left(\lambda I_{\lambda}+\mu I_{\mu}\right) \varepsilon \mathrm{d} V \tag{1.5.12}
\end{equation*}
$$

Note that $\lambda$ and $\mu$ may depend on $(x, y, z)$.

### 1.6 Kinetic energy

Soft robot motion yields kinetic energy. We formulate kinetic energy of soft robots based on kinetic energy density.

Let us formulate the kinetic energy of a one-dimensional soft robot. Let $\rho$ be density of the robot material. Recall that velocity at point $\mathrm{P}(x)$ is given by $\dot{u}=\partial u / \partial t$. Then, kinetic energy density at point $\mathrm{P}(x)$ is formulated as follows:

$$
\begin{equation*}
\frac{1}{2} \rho \dot{u}^{2} \tag{1.6.1}
\end{equation*}
$$

The above quantity has the dimension of energy/volume. Integrating the product of the kinetic energy density and the volume element over one-dimensional region $[0, L$ ] yields kinetic energy. Consequently, kinetic energy of this one-dimensional soft robot is formulated as follows:

$$
\begin{equation*}
T=\int_{0}^{L} \frac{1}{2} \rho \dot{u}^{2} A \mathrm{~d} x=\int_{0}^{L} \frac{1}{2} \rho A\left(\frac{\partial u}{\partial t}\right)^{2} \mathrm{~d} x \tag{1.6.2}
\end{equation*}
$$

Note that $\rho$ and $A$ may depend on $x$.
Let us formulate the kinetic energy of a two-dimensional soft robot. Velocity vector at point $\mathrm{P}(x, y)$ is given by $\dot{\boldsymbol{u}}=[\dot{u}, \dot{v}]^{\top}$. Kinetic energy density is then formulated as follows:

$$
\begin{equation*}
\frac{1}{2} \rho \dot{\boldsymbol{u}}^{\top} \dot{\boldsymbol{u}}=\frac{1}{2} \rho\left(\dot{u}^{2}+\dot{v}^{2}\right) \tag{1.6.3}
\end{equation*}
$$

Integrating the product of the energy density and the volume element over two-dimensional region $S$ yields kinetic energy. Consequently, kinetic energy of this two-dimensional soft robot is formulated as follows:

$$
\begin{equation*}
T=\int_{S} \frac{1}{2} \rho \dot{\boldsymbol{u}}^{\top} \dot{\boldsymbol{u}} h \mathrm{~d} S \tag{1.6.4}
\end{equation*}
$$

Note that $\rho$ and $h$ may depend on $(x, y)$.
Let us formulate the kinetic energy of a three-dimensional soft robot. Velocity vector at point at point $\mathrm{P}(x, y, z)$ is given by $\dot{\boldsymbol{u}}=[\dot{u}, \dot{v}, \dot{w}]^{\top}$. Kinetic energy density then is formulated as follows:

$$
\begin{equation*}
\frac{1}{2} \rho \dot{\boldsymbol{u}}^{\top} \dot{\boldsymbol{u}}=\frac{1}{2} \rho\left(\dot{u}^{2}+\dot{v}^{2}+\dot{w}^{2}\right) \tag{1.6.5}
\end{equation*}
$$

Integrating the product of the energy density and the volume element over three-dimensional region $V$ yields kinetic energy. Consequently, kinetic energy of this three-dimensional soft robot is formulated as follows:

$$
\begin{equation*}
T=\int_{V} \frac{1}{2} \rho \dot{\boldsymbol{u}}^{\top} \dot{\boldsymbol{u}} \mathrm{d} V \tag{1.6.6}
\end{equation*}
$$

Note that density $\rho$ may depend on $(x, y, z)$.

## Problems

1. Gravitational force acts to a two-dimensional soft robot along $y$-axis in its negative direction. Let $g$ be gravitational acceleration. Show that the gravitational potential energy.is formulated as

$$
U_{\text {grav }}=\int_{S} \rho g y h \mathrm{~d} S
$$

Gravitational force acts to a three-dimensional soft robot along $z$-axis in its negative direction. Show that the gravitational potential energy.is formulated as

$$
U_{\text {grav }}=\int_{V} \rho g z \mathrm{~d} V
$$

