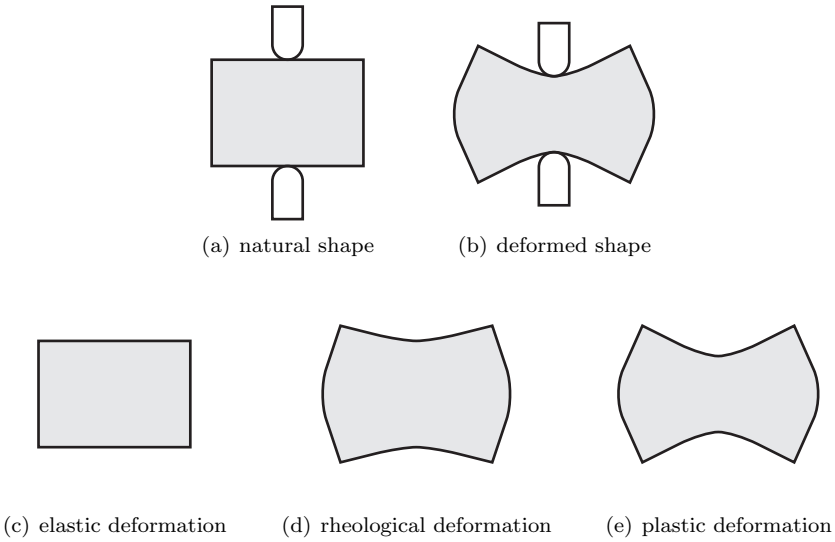
Chapter 5

Deformation Models

5.1 Classification of Deformation

Let us push a soft object of which natural shape is given in Fig. 5.1(a). The object deforms according to the applied force (Fig. 5.1(b)). Let us observe the object shape after the applied force is released. Depending on the shape after the release, object deformation can be roughly classified into three categories in our definition: *elastic* deformation, *plastic* deformation, and *rheological* deformation. Elastic deformation implies that the deformation caused by the applied force is completely reversible (Fig. 5.1(c)). Contrary, plastic deformation implies that the deformation caused by the applied force is completely maintained (Fig. 5.1(e)). Rheological deformation implies that the deformation is partially reversible (Fig. 5.1(d)). Rheological deformation is also referred to as *elastoplastic* deformation.



(a) natural shape (b) deformed shape

(c) elastic deformation (d) rheological deformation (e) plastic deformation

Figure 5.1: Classification of object deformation

5.2 Elementary Models

5.2.1 Elastic model

Elasticity determines the relationship between stress and strain. We can feel the hardness of an object on a table by pushing the object by a finger and observing the surface displacement against the applied force. For example, pushing the surface of a aluminum plate yields little displacement while a rubber plate would be deformed much by the equally applied force. This deformation depends on not only material property but also the object geometry. For example, when a force is applied to an object through a narrow area, the object deforms much. A thicker object deforms much even though the same force is applied. Concept of stress and strain is thus introduced to avoid the effect of the geometry. *Stress* is determined by the applied force divided by the area, through which the force is applied to the object. *Strain* is determined by the displacement divided by the object thickness. Namely, stress is a normalized force applied to a unit area while strain is a normalized displacement. Thus, stress and strain avoid the effect of the geometry, implying that the relationship between stress and strain describes the property of a material itself. Stress-strain relationship determines how a material deforms according to applied stress or strain. Generally, the relationship is nonlinear and time-variant. In the formulation of object deformation, we often assume an ideal relationship: *linear and time-invariant elasticity*, which is represented by a linear elastic model.

Let σ be the stress applied to a material and ε be the strain of the material. Linear elasticity is then described as

$$\sigma = E\varepsilon, \tag{5.2.1}$$

where constant E is referred to as *Young's modulus*. Elastic model is described by a symbol shown in Fig. 5.2(a). Note that unit of σ is $\mathrm{N/m^2}$ or Pa and ε is a dimensionless quantity, implying that unit of Young's modulus is $\mathrm{N/m^2}$ or Pa. Young's modulus determines the hardness of a material itself. For example, Young's modulus of aluminum is about 7.0×10^{10} Pa while Young's modulus of rubber stays around 10^6 Pa, suggesting that aluminum is harder than rubber.

5.2.2 Viscous model

Viscosity determines the relationship between stress and the rate of strain. Let us push a bread dough by a hand. When we push the dough faster, we feel larger force. Pushing the dough slower yields smaller force. Especially, after stopping the pushing, we feel little force. Namely, the force depends on the speed of pushing rather than the displacement of pushing. This suggests a relationship between stress and the rate of strain. Generally, the relationship is nonlinear and time-variant as well. We often assume an ideal relationship: *linear and time-invariant viscosity*, which is represented by a linear viscous element.

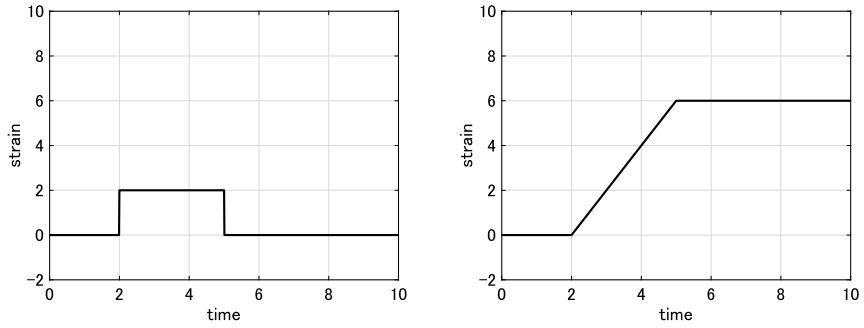
Let σ be the stress applied to a material and ε be the strain of the material. Linear viscosity is then described as

$$\sigma = c\dot{\varepsilon}, \tag{5.2.2}$$

where constant c is referred to as *viscous modulus*. Viscous model is described by a symbol shown in Fig. 5.2(b). Note that unit of σ is $\mathrm{N/m^2}$ or Pa and unit of $\dot{\varepsilon}$ is 1/s, implying that unit of viscous modulus is $\mathrm{Ns/m^2}$ or $\mathrm{Pa \cdot s}$.

(a) elastic element (b) viscous element

Figure 5.2: Elementary models



(a) elastic element (b) viscous element

Figure 5.3: Strain responses of elastic and viscous elements to constant stress. $t_1 = 2.00$, $t_2 = 5.00$, $\sigma_0 = 10.00$, $E = 5.00$, and $c = 5.00$.

5.2.3 Response to constant stress

Let us apply constant stress σ_0 to elastic model during time period $[\,t_1,\, t_2\,]$. Strain of the model is then described as

$$\varepsilon = \begin{cases} 0 & t \le t_1 \\ \sigma_0/E & t \in [\,t_1,\, t_2\,] \\ 0 & t \ge t_2 \end{cases}$$

which is plotted in Fig. 5.3(a). We have non-zero strain σ_0/E while a constant stress is applied during $[\,t_1,\, t_2\,]$. After the stress is released at t_2, we have no strain. Thus, this model can describe elastic deformation. But, discontinuity of strain happens at time t_1 and t_2, which is unacceptable to dynamics. This discontinuity can be eliminated by introducing Voigt model in Section 5.3.1.

Let us apply constant stress σ_0 to viscous model during time period $[\,t_1,\, t_2\,]$. Strain of the model is then described as

$$\varepsilon = \begin{cases} 0 & t \le t_1 \\ (\sigma_0/c)(t - t_1) & t \in [\,t_1,\, t_2\,] \\ (\sigma_0/c)(t_2 - t_1) & t \ge t_2 \end{cases}$$

which is plotted in Fig. 5.3(b). Strain increases constantly at the rate of σ_0/c as long as a constant stress is applied during $[\,t_1,\, t_2\,]$. After the stress is released at t_2, constant strain $(\sigma_0/c)(t_2 - t_1)$ remains. Thus, this model can describe plastic deformation. Note that this strain response is continuous.

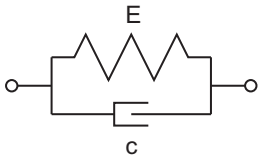
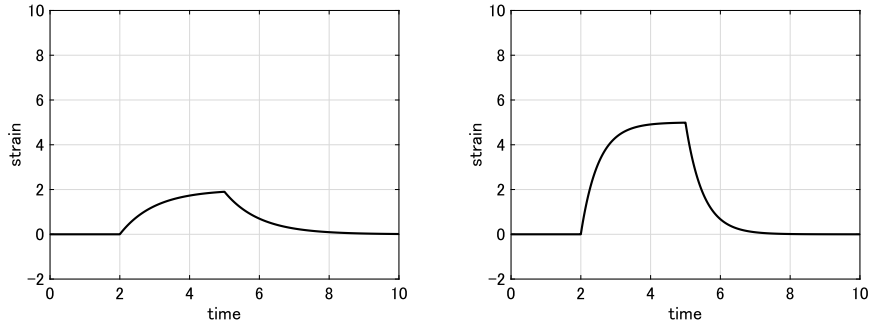


Figure 5.4: Voigt model



(a) $E = 5.00$, $c = 5.00$ (b) $E = 2.00$, $c = 1.00$

Figure 5.5: Strain responses of Voigt model to constant stress. $t_1 = 2.00$, $t_2 = 5.00$, and $\sigma_0 = 10.00$.

5.3 Combined Models

Connecting elastic and viscous models in parallel and serial yields combined models, which can describe complicated relationship between stress and strain.

5.3.1 Voigt model

As mentioned in Section 5.2.3, elastic model causes discontinuity in strain for constant stress. Note that rate of strain diverges when stress applied to the elastic model changes discontinuously. Inserting a viscous model in parallel to the elastic model to prevent the rate of strain from diverging yields *Voigt model* (Fig. 5.4). This model is also referred to as *Kelvin–Voigt model*.

Let σ be the stress applied to Voigt model and ε be the strain of the model. Stress σ is the sum of two stresses: σ^{ela} caused by the elastic model and σ^{vis} caused by the viscous model. Thus, we have the following equations:

$$\sigma = \sigma^{\mathrm{ela}} + \sigma^{\mathrm{vis}}, \qquad \sigma^{\mathrm{ela}} = E\varepsilon, \qquad \sigma^{\mathrm{ela}} = c\dot{\varepsilon}.$$

Eliminating σ^{ela} and σ^{vis} directly yields the stress-strain relationship of Voigt model:

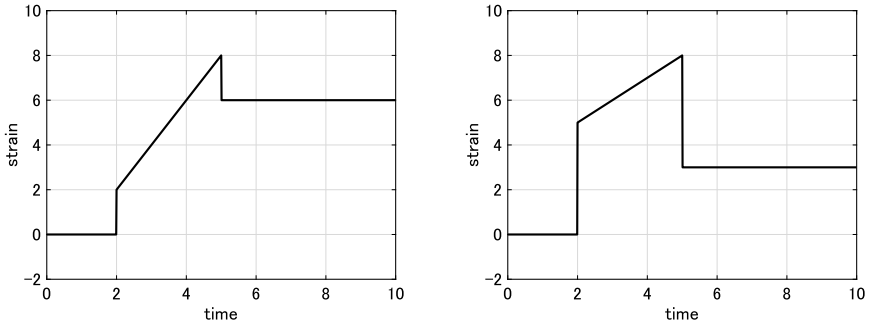
$$\sigma = E\varepsilon + c\dot{\varepsilon}. \tag{5.3.1}$$

When $c = 0$ in the above equation, a Voigt model coincides with an elastic element. When $E = 0$ in the above equation, a Voigt model coincides with a viscous element.

Solving the above ordinary differential equation eq. (5.3.1) under $\varepsilon(0) = 0$, we can describe strain $\varepsilon(t)$ at time t in a convolution form as

$$\varepsilon(t) = \int_0^t \frac{1}{c} e^{-\frac{E}{c}(t-t')} \sigma(t') \, \mathrm{d}t'. \tag{5.3.2}$$

Figure 5.6: Maxwell model



(a) $E = 5.00$, $c = 5.00$ (b) $E = 2.00$, $c = 10.0$

Figure 5.7: Strain responses of Maxwell model to constant stress. $t_1 = 2.00$, $t_2 = 5.00$, and $\sigma_0 = 10.00$.

Let us apply constant stress σ_0 to Voigt model during time period $[\, t_1,\, t_2\,]$. Strain of the model is then described as

$$\varepsilon(t) = \begin{cases} 0 & t \le t_1 \\ (\sigma_0/E)\{1 - e^{-\frac{E}{c}(t-t_1)}\} & t \in [\, t_1,\, t_2\,] \\ (\sigma_0/E)\{1 - e^{-\frac{E}{c}(t_2-t_1)}\}e^{-\frac{E}{c}(t-t_2)} & t \ge t_2 \end{cases} \tag{5.3.3}$$

(see Problem 1). From the above equation, we find that strain converges to stationary strain σ_0/E as long as constant stress is applied to the model. Additionally, ratio E/c determines how fast the strain changes. Figure 5.5 shows two examples of the strain response. Figure 5.3.1 shows the response at ratio of 1.00 while Fig. 5.3.1 shows the response at ratio of 2.00.

5.3.2 Maxwell model

Elastic model completely reverses while viscous model does not reverse at all. Connecting an elastic and viscous models in serial to describe partially reversible deformation yields *Maxwell model* (Fig. 5.6). Let E be Young's modulus, which represents the elastic model, and c be viscous modulus, which characterizes the viscous model. Let ε be strain of the Maxwell model and σ be stress applied to the model. Strain ε coincides to the sum of two strains: strain $\varepsilon^{\mathrm{ela}}$ of the elastic model and strain $\varepsilon^{\mathrm{vis}}$ of the viscous model. Stress σ is equal to the stress caused by the elastic element as well as the stress caused by the viscous element. That is,

$$\varepsilon = \varepsilon^{\mathrm{ela}} + \varepsilon^{\mathrm{vis}}, \qquad \sigma = E\varepsilon^{\mathrm{ela}}, \qquad \sigma = c\dot{\varepsilon}^{\mathrm{vis}}. \tag{5.3.4}$$

From the above equations, we have the following first order differential equation:

$$\dot{\sigma} + \frac{E}{c}\sigma = E\dot{\varepsilon}. \tag{5.3.5}$$

Solving the above differential equation under $\sigma(0) = 0$, stress at time t is described as follows in a convolution form:

$$\sigma(t) = \int_0^t E e^{-\frac{E}{c}(t-t')} \dot{\varepsilon}(t') \, \mathrm{d}t'. \tag{5.3.6}$$

When $c \to \infty$ in the above equation, a Maxwell model coincides with an elastic element. When $E \to \infty$ in the above equation, a Maxwell model coincides with a viscous element (see Problem 3). In general, we have the following convolution form:

$$\sigma(t) = \int_0^t r(t-t') \, \dot{\varepsilon}(t') \, \mathrm{d}t'. \tag{5.3.7}$$

In Maxwell model

$$r(t-t') = E e^{-\frac{E}{c}(t-t')}. \tag{5.3.8}$$

Function $r(t-t')$ is referred to as a *relaxation function*.

Let us apply constant stress σ_0 to Maxwell model during time period $[\, t_1, \, t_2 \,]$. Strain of the model is then described as

$$\varepsilon(t) = \begin{cases} 0 & t \le t_1 \\ (\sigma_0/E) + (\sigma_0/c)(t - t_1) & t \in [\, t_1, \, t_2 \,] \\ (\sigma_0/c)(t_2 - t_1) & t \ge t_2 \end{cases} \tag{5.3.9}$$

which is plotted in Fig. 5.7 (see Problem 2).

Let us reformulate eq. (5.3.6) using Laplace transform. Let $\sigma(s)$, $\varepsilon(s)$, $\varepsilon^{\mathrm{ela}}(s)$, and $\varepsilon_{\mathrm{vis}}(s)$ are Laplace transforms of $\sigma(t)$, $\varepsilon(t)$, $\varepsilon^{\mathrm{ela}}(t)$, and $\varepsilon^{\mathrm{vis}}(t)$. Then, we have

$$\varepsilon(s) = \varepsilon^{\mathrm{ela}}(s) + \varepsilon^{\mathrm{vis}}(s), \qquad \sigma(s) = E\varepsilon^{\mathrm{ela}}(s), \qquad \sigma(s) = cs\varepsilon^{\mathrm{vis}}(s).$$

From the above equations, we have

$$\sigma(s) = \frac{E}{s + E/c} \, s\varepsilon(s).$$

Applying the inverse Laplace transform to the above equation successfully yields eq. (5.3.6).

5.3.3 Three-element models

As mentioned in Section 5.3.2, Maxwell model causes discontinuity in strain for constant stress. This originates from the elastic element in Maxwell model. Replacing this elastic model by Voigt model to avoid discontinuity in strain yields *three-element model* (Fig. 5.8). Three-element model consists of a Voigt element and a viscous element connected in serial. Let E and c_1 be Young's modulus and viscous modulus of the Voigt element, and c_2 be viscous modulus of the viscous element. Let $\varepsilon^{\mathrm{voigt}}$ and $\varepsilon^{\mathrm{vis}}$ be strains at the Voigt and viscous elements. Let ε be strain of the three-element model and σ be stress applied to the model. Strain ε coincides to the sum of strains of the two elements. Stress σ is equal to the stress caused by the Voigt element as well as the stress caused by the viscous element. That is,

$$\varepsilon = \varepsilon^{\mathrm{voigt}} + \varepsilon^{\mathrm{vis}}, \qquad \sigma = E\varepsilon^{\mathrm{voigt}} + c_1\dot{\varepsilon}^{\mathrm{voigt}}, \qquad \sigma = c_2\dot{\varepsilon}^{\mathrm{vis}}. \tag{5.3.10}$$

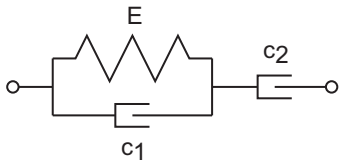
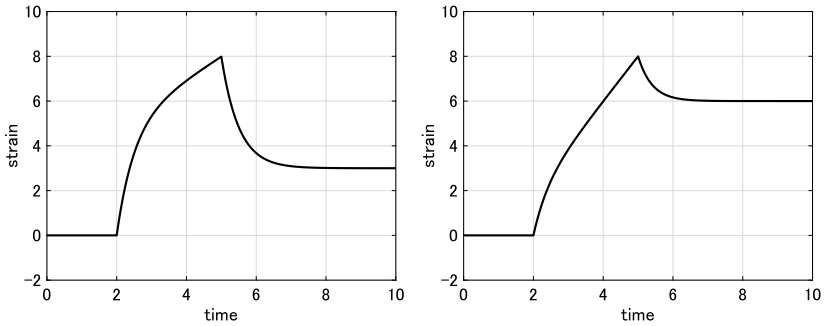


Figure 5.8: Three-element model consisting of Voigt and viscous elements



(a) $E = 2.00$, $c_1 = 1.00$, $c_2 = 10.00$ (b) $E = 5.00$, $c_1 = 2.00$, $c_2 = 5.00$

Figure 5.9: Strain responses of three-element model to constant stress. $t_1 = 2.00$, $t_2 = 5.00$, and $\sigma_0 = 10.00$.

From the above equations, we have the following first order differential equation on stress σ:

$$\dot{\sigma} + \frac{E}{c_1 + c_2}\sigma = \frac{Ec_2}{c_1 + c_2}\dot{\varepsilon} + \frac{c_1 c_2}{c_1 + c_2}\ddot{\varepsilon}. \tag{5.3.11}$$

Solving the above differential equation, stress at time t is described in a convolution form given in eq. (5.3.7), with relaxation function

$$r(t - t') = \frac{c_2}{c_1 + c_2} e^{-\frac{E}{c_1 + c_2}(t - t')} \left(E + c_1 \frac{\mathrm{d}}{\mathrm{d}t} \right). \tag{5.3.12}$$

Note that this relaxation function involves operator $\mathrm{d}/\mathrm{d}t$.

Let us apply constant stress σ_0 to three-element model during time period $[\, t_1,\, t_2\,]$. Solving eq. (5.3.10), we directly have

$$\varepsilon^{\mathrm{voigt}}(t) = \int_0^t \frac{1}{c_1} e^{-\frac{E}{c_1}(t - t')} \sigma(t')\, \mathrm{d}t', \qquad \varepsilon^{\mathrm{vis}}(t) = \int_0^t \frac{1}{c_2} \sigma(t')\, \mathrm{d}t',$$

which implies that strain caused by the three-element model is a simple sum of strain caused by a Voigt model (see Section 5.3.1) and strain caused by a viscous model (see Section 5.2.2). Figure 5.9 shows two examples of the strain response.

Let us reformulate eq. (5.3.12) using Laplace transform. Let $\sigma(s)$, $\varepsilon(s)$, $\varepsilon^{\mathrm{voigt}}(s)$, and $\varepsilon_{\mathrm{vis}}(s)$ are Laplace transforms of $\sigma(t)$, $\varepsilon(t)$, $\varepsilon^{\mathrm{voigt}}(t)$, and $\varepsilon^{\mathrm{vis}}(t)$. Then, we have

$$\varepsilon(s) = \varepsilon^{\mathrm{voigt}}(s) + \varepsilon^{\mathrm{vis}}(s), \qquad \sigma(s) = E\varepsilon^{\mathrm{voigt}}(s) + c_1 s \varepsilon^{\mathrm{voigt}}(s), \qquad \sigma(s) = c_2 s \varepsilon^{\mathrm{vis}}(s).$$

From the above equations, we have

$$\sigma(s) = \left(\frac{c_2}{c_1 + c_2} \right) \frac{1}{s + E/(c_1 + c_2)} \; (E + c_1 s)\, s\varepsilon(s).$$

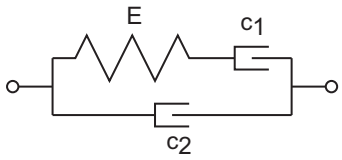


Figure 5.10: Three-element model consisting of Maxwell and viscous elements

Applying the inverse Laplace transform to the above equation successfully yields eq. (5.3.12).

Let us construct a model consisting of a Maxwell and a viscous elements connected in parallel (Fig. 5.10). Let σ^{maxwell} and σ^{vis} be stresses applied to the Maxwell and viscous elements. Stress σ coincides to the sum of the two stresses. Strain ε is equal to the strain of the Maxwell element as well as the strain of the viscous element. That is,

$$\sigma = \sigma^{\text{maxwell}} + \sigma^{\text{vis}}, \qquad \dot{\sigma}^{\text{maxwell}} + \frac{E}{c_1}\sigma^{\text{maxwell}} = E\dot{\varepsilon}, \qquad \sigma^{\text{vis}} = c_2\dot{\varepsilon}.$$

From the above equations, we have the following first order differential equation on stress σ:

$$\dot{\sigma} + \frac{E}{c_1}\sigma = \frac{E(c_1+c_2)}{c_1}\dot{\varepsilon} + c_2\ddot{\varepsilon}. \tag{5.3.13}$$

Solving the above differential equation, stress at time t is described in a convolution form given in eq. (5.3.7), with relaxation function

$$r(t-t') = \frac{c_1+c_2}{c_1}e^{-\frac{E}{c_1}(t-t')}\left(E + \frac{c_1c_2}{c_1+c_2}\frac{\mathrm{d}}{\mathrm{d}t}\right).$$

Eqs. (5.3.11)(5.3.13) can be converted each other (see Problem 4), suggesting that models in Fig. 5.8 and in Fig. 5.10 are equivalent each other.

5.3.4 Standard linear solid models

Replacing viscous element c_2 in three-element models by an elastic element, we have *standard linear solid models*, which are often applied to describe deformation of solids. Stress-strain relationship of the model shown in Fig. 5.11(a) is formulated as

$$\dot{\sigma} + \frac{E_1+E_2}{c}\sigma = \frac{E_1E_2}{c}\varepsilon + E_2\dot{\varepsilon}. \tag{5.3.14}$$

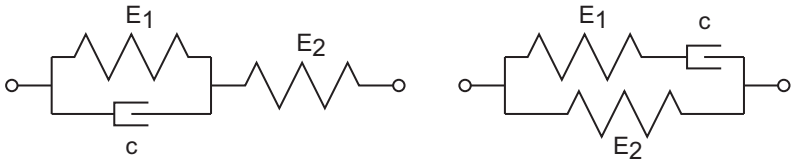
Stress-strain relationship of the model shown in Fig. 5.11(b) is formulated as

$$\dot{\sigma} + \frac{E_1}{c}\sigma = \frac{E_1E_2}{c}\varepsilon + (E_1+E_2)\dot{\varepsilon}. \tag{5.3.15}$$

Eqs. (5.3.14)(5.3.15) can be converted each other (see Problem 5).

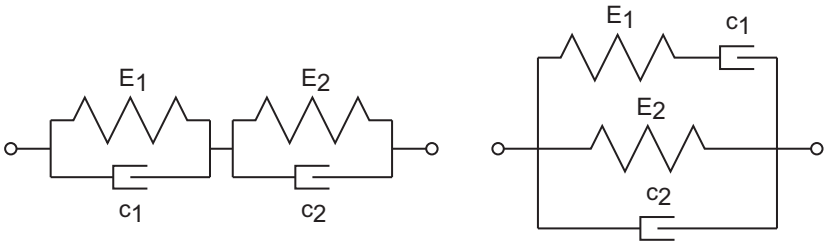
The above models may cause discontinuity in strain. We will apply standard linear solid models with viscous elements (Fig. 5.12) to avoid such discontinuity. Stress-strain relationship of the model shown in Fig. 5.12(a) is formulated as

$$\dot{\sigma} + \frac{E_1+E_2}{c_1+c_2}\sigma = \frac{E_1E_2}{c_1+c_2}\varepsilon + \frac{E_1c_2+E_2c_1}{c_1+c_2}\dot{\varepsilon} + \frac{c_1c_2}{c_1+c_2}\ddot{\varepsilon}. \tag{5.3.16}$$



(a) serial (b) parallel

Figure 5.11: Standard linear solid models



(a) serial (b) parallel

Figure 5.12: Standard linear solid models with viscous elements

Note that when $c_2 = 0$, the above equation coincides with eq. (5.3.14). Stress-strain relationship of the model shown in Fig. 5.12(b) is formulated as

$$\dot{\sigma} + \frac{E_1}{c_1}\sigma = \frac{E_1 E_2}{c_1}\varepsilon + \left(E_1 + E_2 + \frac{c_2}{c_1}E_1\right)\dot{\varepsilon} + c_2\ddot{\varepsilon}. \tag{5.3.17}$$

Note that when $c_2 = 0$, the above equation coincides with eq. (5.3.15).

5.4 Serial and Parallel Models

5.4.1 Serial elastic model

Let us generalize the model shown in Fig. 5.12(a). This model consists of two Voigt models connected in serial. Let us connect n Voigt models in serial, as shown in Fig. 5.13. Let ε_k be the strain of the k-th Voigt element, which is characterized by Young's modulus E_k and viscous modulus c_k. Stress-strain relationship of the k-th Voigt element is then given as

$$\sigma = E_k\varepsilon_k + c_k\dot{\varepsilon}_k, \quad (k = 1, 2, \cdots, n).$$

Total strain is the sum of strains of individual Voigt elements:

$$\varepsilon = \sum_{k=1}^{n} \varepsilon_k.$$

Applying Laplace transform to the above equations, we have

$$\sigma(s) = (E_k + c_k s)\varepsilon_k(s), \quad (k = 1, 2, \cdots, n),$$
$$\varepsilon(s) = \sum_{k=1}^{n} \varepsilon_k(s) = \left\{\sum_{k=1}^{n} \frac{1}{c_k s + E_k}\right\}\sigma(s).$$

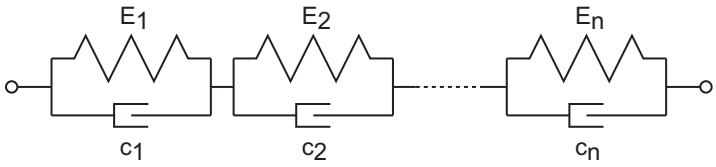


Figure 5.13: Serial elastic model

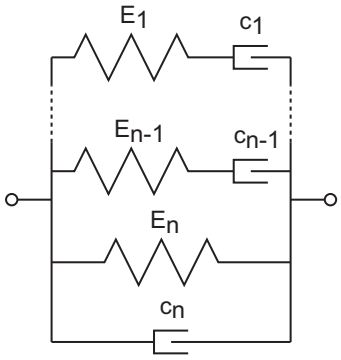


Figure 5.14: Parallel elastic model

Introducing

$$\sum_{k=1}^{n} \frac{1}{c_k s + E_k} \triangleq \frac{B_{n-1}s^{n-1} + \cdots + B_1 s + B_0}{A_n s^n + A_{n-1}s^{n-1} + \cdots + A_1 s + A_0}, \tag{5.4.1}$$

we have the stress-strain relationship:

$$\left(B_{n-1}s^{n-1} + \cdots + B_1 s + B_0\right)\sigma(s) = \left(A_n s^n + \cdots + A_1 s + A_0\right)\varepsilon(s),$$

which yields

$$\sum_{j=0}^{n-1} B_j \sigma^{(j)} = \sum_{j=0}^{n} A_j \varepsilon^{(j)}.$$

Namely, we have an ordinary differential equation of the $(n-1)$-th order with respect to σ while of the n-th order with respect to ε.

5.4.2 Parallel elastic model

Let us generalize the model shown in Fig. 5.12(b). This model consists of a Maxwell model and a Voigt model connected in parallel. Let us connect $(n-1)$ Maxwell models and a Voigt model in parallel, as shown in Fig. 5.14. Let σ_k be the stress applied to the k-th Maxwell model, which is characterized by Young's modulus E_k and viscous modulus c_k. Stress-strain relationship of the k-th Maxwell element is then given as

$$\dot{\sigma}_k + \frac{E_k}{c_k}\sigma_k = E_k \dot{\varepsilon}, \quad (k = 1, 2, \cdots, n-1).$$

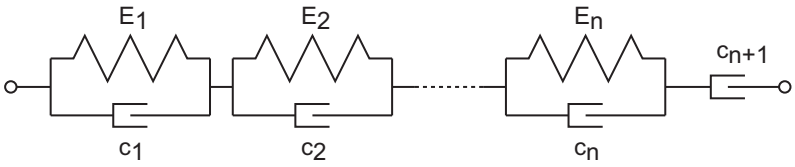


Figure 5.15: Serial rheological model

Total stress is the sum of stresses of individual Maxwell elements and a Voigt element:

$$\sigma = \sum_{k=1}^{n-1} \sigma_k + (E_n \varepsilon + c_n \dot{\varepsilon}).$$

Applying Laplace transform to the above equations, we have

$$\left(s + \frac{E_k}{c_k}\right) \sigma_k(s) = E_k s \varepsilon(s), \quad (k = 1, 2, \cdots, n-1),$$
$$\sigma(s) = \sum_{k=1}^{n-1} \sigma_k(s) + (E_n + c_n s)\varepsilon(s) = \left\{ \sum_{k=1}^{n-1} \frac{c_k E_k s}{c_k s + E_k} + E_n + c_n s \right\} \varepsilon(s).$$

Introducing

$$\sum_{k=1}^{n-1} \frac{c_k E_k s}{c_k s + E_k} + E_n + c_n s \triangleq \frac{C_n s^n + C_{n-1} s^{n-1} + \cdots + C_1 s + C_0}{D_{n-1} s^{n-1} + \cdots + D_1 s + D_0}, \tag{5.4.2}$$

we have the stress-strain relationship:

$$\left(D_{n-1} s^{n-1} + \cdots + D_1 s + D_0\right) \sigma(s) = \left(C_n s^n + \cdots + C_1 s + C_0\right) \varepsilon(s),$$

which yields

$$\sum_{j=0}^{n-1} D_j \sigma^{(j)} = \sum_{j=0}^{n} C_j \varepsilon^{(j)}.$$

Namely, we have an ordinary differential equation of the $(n-1)$-th order with respect to σ while of the n-th order with respect to ε.

5.4.3 Serial rheological model

Let us generalize the model shown in Fig. 5.8. This model consists of a Voigt model and a viscous model connected in serial. Let us connect n Voigt models and a viscous model in serial, as shown in Fig. 5.15. Let ε_k be the strain of the k-th Voigt element, which is characterized by Young's modulus E_k and viscous modulus c_k. Let ε_{n+1} be the strain of the viscous element specified by viscous modulus c_{n+1}. Stress-strain relationships are then given as

$$\sigma = E_k \varepsilon_k + c_k \dot{\varepsilon}_k, \quad (k = 1, 2, \cdots, n), \qquad \sigma = c_{n+1} \dot{\varepsilon}_{n+1}.$$

Total strain is the sum of strains of individual elements:

$$\varepsilon = \sum_{k=1}^{n} \varepsilon_k + \varepsilon_{n+1}.$$

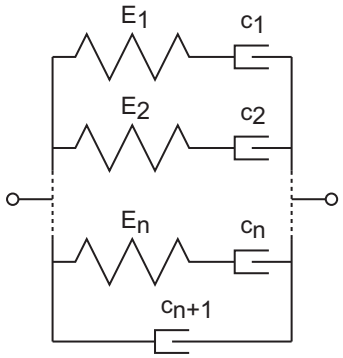


Figure 5.16: Parallel rheological model

Applying Laplace transform to the above equations, we have

$$\sigma(s) = (E_k + c_k s)\varepsilon_k(s), \quad (k = 1, 2, \cdots, n),$$
$$\sigma(s) = c_{n+1} s \varepsilon_{n+1}(s),$$
$$\varepsilon(s) = \sum_{k=1}^{n} \varepsilon_k(s) + \varepsilon_{n+1}(s) = \left\{ \sum_{k=1}^{n} \frac{1}{c_k s + E_k} + \frac{1}{c_{n+1} s} \right\} \sigma(s).$$

Introducing

$$\sum_{k=1}^{n} \frac{1}{c_k s + E_k} + \frac{1}{c_{n+1} s} \triangleq \frac{B_n s^n + B_{n-1} s^{n-1} + \cdots + B_1 s + B_0}{s(A_n s^n + A_{n-1} s^{n-1} + \cdots + A_1 s + A_0)}, \tag{5.4.3}$$

we have the stress-strain relationship:

$$(B_n s^n + \cdots + B_1 s + B_0)\, \sigma(s) = (A_n s^n + + \cdots + A_1 s + A_0)\, s\varepsilon(s),$$

which yields

$$\sum_{j=0}^{n} B_j \sigma^{(j)} = \sum_{j=0}^{n} A_j \varepsilon^{(j+1)}.$$

Namely, we have an ordinary differential equation of the n-th order with respect to σ while of the n-th order with respect to $\dot{\varepsilon}$.

5.4.4 Parallel rheological model

Let us generalize the model shown in Fig. 5.10. This model consists of a Maxwell model and a viscous model connected in parallel. Let us connect n Maxwell models and a viscous model in parallel, as shown in Fig. 5.16. Let σ_k be the stress applied to the k-th Maxwell model, which is characterized by Young's modulus E_k and viscous modulus c_k. Let σ_{n+1} be the stress applied to the viscous model specified by viscous modulus c_{n+1}. Stress-strain relationships are then given as

$$\dot{\sigma}_k + \frac{E_k}{c_k}\sigma_k = E_k \dot{\varepsilon}, \quad (k = 1, 2, \cdots, n), \qquad \sigma_{n+1} = c_{n+1}\dot{\varepsilon}.$$

Total strain is the sum of strains of individual elements:

$$\sigma = \sum_{k=1}^{n} \sigma_k + \sigma_{n+1}.$$

Applying Laplace transform to the above equations, we have

$$\begin{aligned}
\left(s + \frac{E_k}{c_k}\right) \sigma_k(s) &= E_k s\varepsilon(s), \quad (k = 1, 2, \cdots, n), \\
\sigma_{n+1}(s) &= c_{n+1} s\varepsilon(s), \\
\sigma(s) = \sum_{k=1}^{n} \sigma_k(s) + \sigma_{n+1}(s) &= \left\{ \sum_{k=1}^{n} \frac{c_k E_k s}{c_k s + E_k} + c_{n+1} s \right\} \varepsilon(s).
\end{aligned}$$

Introducing

$$\sum_{k=1}^{n} \frac{c_k E_k s}{c_k s + E_k} + c_{n+1} s \triangleq \frac{s(C_n s^n + C_{n-1} s^{n-1} + \cdots + C_1 s + C_0)}{D_n s^n + D_{n-1} s^{n-1} + \cdots + D_1 s + D_0}, \tag{5.4.4}$$

we have the stress-strain relationship:

$$(D_n s^n + \cdots + D_1 s + D_0)\, \sigma(s) = (C_n s^n + \cdots + C_1 s + C_0)\, s\varepsilon(s),$$

which yields

$$\sum_{j=0}^{n} D_j \sigma^{(j)} = \sum_{j=0}^{n} C_j \varepsilon^{(j+1)}.$$

Namely, we have an ordinary differential equation of the n-th order with respect to σ while of the n-th order with respect to $\dot{\varepsilon}$.

Problems

1. Show eq. (5.3.3).
2. Show eq. (5.3.9).
3. Show a Maxwell model coincides with an elastic element when $c \to \infty$ and a Maxwell model coincides with a viscous element when $E \to \infty$.
4. Show that eqs. (5.3.11)(5.3.13) can be converted each other.
5. Show that eqs. (5.3.14)(5.3.15) can be converted each other.