## Chapter 6

## Inelastic Deformation

### 6.1 Multi-dimensional inelastic deformation models

We extend one-dimensional inelastic deformation models (Chapter 5) to multi-dimensional inelastic deformation models under assumption that materials exhibit isotropic deformation.

Recall that the stress-strain relationship of an elastic material can be specified by a constant $E$. In addition, two- or three-dimensional isotropic elastic deformation can be formulated as follows:

$$
\begin{equation*}
\boldsymbol{\sigma}=\left(\lambda I_{\lambda}+\mu I_{\mu}\right) \boldsymbol{\varepsilon} \tag{6.1.1}
\end{equation*}
$$

where $\lambda$ and $\mu$ denote Lamé's constants and matrices $I_{\lambda}$ and $I_{\mu}$ originate from the isotropy of the material. Elasticity can be specified by two constants: $\lambda$ and $\mu$. These constants determine normal elasticity and shear elasticity.

Recall that the stress-strain relationship of a viscoelastic material can be specified by an operator: $E+c \mathrm{~d} / \mathrm{d} t$. From the above observation, replacing two elastic constants in eq. (6.1.1) by two viscoelastic operators yields two- or three-dimensional isotropic viscoelastic deformation as follows:

$$
\begin{equation*}
\boldsymbol{\sigma}=\left(\lambda I_{\lambda}+\mu I_{\mu}\right) \boldsymbol{\varepsilon} \tag{6.1.2}
\end{equation*}
$$

where

$$
\lambda=\lambda^{\mathrm{ela}}+\lambda^{\mathrm{vis}} \frac{\mathrm{~d}}{\mathrm{~d} t}, \quad \mu=\mu^{\mathrm{ela}}+\mu^{\mathrm{vis}} \frac{\mathrm{~d}}{\mathrm{~d} t} .
$$

Two constants $\lambda^{\text {ela }}$ and $\mu^{\text {ela }}$ specify elasticity of the material while $\lambda^{\text {vis }}$ and $\mu^{\text {vis }}$ describe its viscosity.

Recall that the stress-strain relationship of rheological models can be specified by a convolution form eq. (5.3.7) with a relaxation function $r\left(t^{\prime}-t\right)$. Then, two- or three-dimensional Maxwell deformation can be described as follows:

$$
\begin{equation*}
\boldsymbol{\sigma}(t)=\int_{0}^{t} R\left(t-t^{\prime}\right) \dot{\varepsilon}\left(t^{\prime}\right) \mathrm{d} t^{\prime} \tag{6.1.3}
\end{equation*}
$$

where $3 \times 3$ matrix $R\left(t-t^{\prime}\right)$ is referred to as a relaxation matrix, which determines the nature of a two- or three-dimensional rheological deformation. Replacing two elastic constants in eq. (6.1.1) by two relaxation functions yields a relaxation matrix of two- or three-dimensional isotropic rheological deformation:

$$
\begin{equation*}
R\left(t-t^{\prime}\right)=r_{\lambda}\left(t-t^{\prime}\right) I_{\lambda}+r_{\mu}\left(t-t^{\prime}\right) I_{\mu} . \tag{6.1.4}
\end{equation*}
$$

Recalling relaxation function eq. (5.3.8) in one-dimensional Maxwell model, two relaxation functions $r_{\lambda}\left(t-t^{\prime}\right)$ and $r_{\mu}\left(t-t^{\prime}\right)$ in two- or three-dimensional isotropic Maxwell model turn into:

$$
\begin{align*}
& r_{\lambda}\left(t-t^{\prime}\right)=\lambda^{\text {ela }} \exp \left\{-\frac{\lambda^{\text {ela }}}{\lambda^{\text {vis }}}\left(t-t^{\prime}\right)\right\},  \tag{6.1.5a}\\
& r_{\mu}\left(t-t^{\prime}\right)=\mu^{\text {ela }} \exp \left\{-\frac{\mu^{\text {ela }}}{\mu^{\text {vis }}}\left(t-t^{\prime}\right)\right\} . \tag{6.1.5b}
\end{align*}
$$

Constants $\lambda^{\text {ela }}$ and $\mu^{\text {ela }}$ specify elasticity of the material while $\lambda^{\text {vis }}$ and $\mu^{\text {vis }}$ describe its plasticity.

Recalling relaxation function eq. (5.3.12) in one-dimensional three-element model, two relaxation functions $r_{\lambda}\left(t-t^{\prime}\right)$ and $r_{\mu}\left(t-t^{\prime}\right)$ in two- or three-dimensional isotropic threeelement model turn into:

$$
\begin{align*}
& r_{\lambda}\left(t-t^{\prime}\right)=\frac{\lambda_{2}^{\text {vis }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}} \exp \left\{-\frac{\lambda^{\text {ela }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}}\left(t-t^{\prime}\right)\right\}\left(\lambda^{\text {ela }}+\lambda_{1}^{\text {vis }} \frac{\mathrm{d}}{\mathrm{~d} t}\right)  \tag{6.1.6a}\\
& r_{\mu}\left(t-t^{\prime}\right)=\frac{\mu_{2}^{\text {vis }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}} \exp \left\{-\frac{\mu^{\text {ela }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}}\left(t-t^{\prime}\right)\right\}\left(\mu^{\text {ela }}+\mu_{1}^{\text {vis }} \frac{\mathrm{d}}{\mathrm{~d} t}\right) . \tag{6.1.6b}
\end{align*}
$$

Constants $\lambda^{\text {ela }}$ and $\mu^{\text {ela }}$ specify elasticity of the material, $\lambda_{1}^{\text {vis }}$ and $\mu_{1}^{\text {vis }}$ describe its viscosity, and $\lambda_{2}^{\text {vis }}$ and $\mu_{2}^{\text {vis }}$ show its plasticity.

### 6.2 Nodal forces in inelastic deformation

The stress-strain relationship can be converted into a relationship between a set of forces applied to nodal points and a set of displacements of the points. A set of elastic forces applied to nodal points is given by

$$
\text { elastic force }=-\left(\lambda J_{\lambda}+\mu J_{\mu}\right) \boldsymbol{u}_{N},
$$

where $J_{\lambda}$ and $J_{\mu}$ are geometric matrices determined by object coordinate components of nodal points. The above equation suggests that replacing $I_{\lambda} \varepsilon$ by $J_{\lambda} \boldsymbol{u}_{\mathrm{N}}$ and $I_{\mu} \varepsilon$ by $J_{\mu} \boldsymbol{u}_{\mathrm{N}}$ in the stress-strain relationship eq. (6.1.1) of an elastic material yields the elastic force set.

From the above observation, replacing $I_{\lambda} \varepsilon$ by $J_{\lambda} \boldsymbol{u}_{\mathrm{N}}$ and $I_{\mu} \varepsilon$ by $J_{\mu} \boldsymbol{u}_{\mathrm{N}}$ in the stress-strain relationship eq. (6.1.2) of a viscoelastic material yields a set of viscoelastic forces applied to nodal points as follows:

$$
\begin{equation*}
\text { viscoelastic force }=-\left(\lambda^{\text {ela }} J_{\lambda}+\mu^{\text {ela }} J_{\mu}\right) \boldsymbol{u}_{\mathrm{N}}-\left(\lambda^{\mathrm{vis}} J_{\lambda}+\mu^{\text {vis }} J_{\mu}\right) \dot{\boldsymbol{u}}_{\mathrm{N}} . \tag{6.2.1}
\end{equation*}
$$

Replacing $I_{\lambda} \varepsilon$ by $J_{\lambda} \boldsymbol{u}_{\mathrm{N}}$ and $I_{\mu} \varepsilon$ by $J_{\mu} \boldsymbol{u}_{\mathrm{N}}$ in the stress-strain relationship of a rheological material given in a convolution form eq. (5.3.7) yields a set of rheological forces applied to nodal points as follows:

$$
\begin{equation*}
\text { rheological force }=-J_{\lambda} \int_{0}^{t} r_{\lambda}\left(t-t^{\prime}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}\left(t^{\prime}\right) \mathrm{d} t^{\prime}-J_{\mu} \int_{0}^{t} r_{\mu}\left(t-t^{\prime}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}\left(t^{\prime}\right) \mathrm{d} t^{\prime} \tag{6.2.2}
\end{equation*}
$$

Introducing

$$
\begin{align*}
& \boldsymbol{f}_{\lambda}(t)=\int_{0}^{t} r_{\lambda}\left(t-t^{\prime}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}\left(t^{\prime}\right) \mathrm{d} t^{\prime}  \tag{6.2.3a}\\
& \boldsymbol{f}_{\mu}(t)=\int_{0}^{t} r_{\mu}\left(t-t^{\prime}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}\left(t^{\prime}\right) \mathrm{d} t^{\prime} \tag{6.2.3b}
\end{align*}
$$

a set of rheological forces applied to nodal points is simply described as

$$
\begin{equation*}
\text { rheological force }=-J_{\lambda} \boldsymbol{f}_{\lambda}(t)-J_{\mu} \boldsymbol{f}_{\mu}(t) . \tag{6.2.4}
\end{equation*}
$$

Recalling that relaxation functions in two- or three-dimensional isotropic Maxwell model are given by eqs. (6.1.5a) (6.1.5b), we find

$$
\begin{aligned}
& \boldsymbol{f}_{\lambda}(t)=\int_{0}^{t} \lambda^{\text {ela }} \exp \left\{-\frac{\lambda^{\text {ela }}}{\lambda^{\text {vis }}}\left(t-t^{\prime}\right)\right\} \dot{\boldsymbol{u}}_{\mathrm{N}}\left(t^{\prime}\right) \mathrm{d} t^{\prime} \\
& \boldsymbol{f}_{\mu}(t)=\int_{0}^{t} \mu^{\mathrm{ela}} \exp \left\{-\frac{\mu^{\mathrm{ela}}}{\mu^{\text {vis }}}\left(t-t^{\prime}\right)\right\} \dot{\boldsymbol{u}}_{\mathrm{N}}\left(t^{\prime}\right) \mathrm{d} t^{\prime} .
\end{aligned}
$$

Differentiating that above two equations, we have

$$
\begin{gather*}
\dot{\boldsymbol{f}}_{\lambda}=-\frac{\lambda^{\mathrm{ela}}}{\lambda^{\text {vis }}} \boldsymbol{f}_{\lambda}+\lambda^{\mathrm{ela}} \dot{\boldsymbol{u}}_{\mathrm{N}}  \tag{6.2.5a}\\
\dot{\boldsymbol{f}}_{\mu}=-\frac{\mu^{\mathrm{ela}}}{\mu^{\mathrm{vis}}} \boldsymbol{f}_{\mu}+\mu^{\mathrm{ela}} \dot{\boldsymbol{u}}_{\mathrm{N}} \tag{6.2.5b}
\end{gather*}
$$

Note that the above two ordinary differential equations have the same form as eq. (5.3.5). Namely, replacing $\sigma, \varepsilon, E$, and $c$ in eq. (5.3.5) by $\boldsymbol{f}_{\lambda}, \boldsymbol{u}_{\mathrm{N}}, \lambda^{\text {ela }}$, and $\lambda^{\text {vis }}$ yields eq. (6.2.5a). Similarly, replacing $\sigma, \varepsilon, E$, and $c$ in eq. (5.3.5) by $\boldsymbol{f}_{\mu}, \boldsymbol{u}_{\mathrm{N}}, \mu^{\text {ela }}$, and $\mu^{\text {vis }}$ yields eq. (6.2.5b). Consequently, stress-strain relationship yields two ordinary differential equations with respect to $\boldsymbol{f}_{\lambda}$ and $\boldsymbol{f}_{\mu}$. Then, nodal force vector is given by $-J_{\lambda} \boldsymbol{f}_{\lambda}-J_{\mu} \boldsymbol{f}_{\mu}$.

Recalling that relaxation functions in two- or three-dimensional isotropic three-element model are given by eqs. (6.1.6a)(6.1.6b), we find

$$
\begin{aligned}
& \boldsymbol{f}_{\lambda}(t)=\int_{0}^{t} \frac{\lambda_{2}^{\text {vis }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}} \exp \left\{-\frac{\lambda^{\text {ela }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}}\left(t-t^{\prime}\right)\right\}\left(\lambda^{\text {ela }}+\lambda_{1}^{\text {vis }} \frac{\mathrm{d}}{\mathrm{~d} t}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}\left(t^{\prime}\right) \mathrm{d} t^{\prime} \\
& \boldsymbol{f}_{\mu}(t)=\int_{0}^{t} \frac{\mu_{2}^{\text {vis }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}} \exp \left\{-\frac{\mu^{\text {ela }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}}\left(t-t^{\prime}\right)\right\}\left(\mu^{\text {ela }}+\mu_{1}^{\text {vis }} \frac{\mathrm{d}}{\mathrm{~d} t}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}\left(t^{\prime}\right) \mathrm{d} t^{\prime} .
\end{aligned}
$$

Differentiating that above two equations, we have

$$
\begin{align*}
& \dot{\boldsymbol{f}}_{\lambda}=-\frac{\lambda^{\text {ela }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}} \boldsymbol{f}_{\lambda}+\frac{\lambda^{\text {ela }} \lambda_{2}^{\text {vis }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}} \dot{\boldsymbol{u}}_{\mathrm{N}}+\frac{\lambda_{1}^{\text {vis }} \lambda_{2}^{\text {vis }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}} \ddot{\boldsymbol{u}}_{\mathrm{N}},  \tag{6.2.6a}\\
& \dot{\boldsymbol{f}}_{\mu}=-\frac{\mu^{\text {ela }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}} \boldsymbol{f}_{\mu}+\frac{\mu^{\text {ela }} \mu_{2}^{\text {vis }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}} \dot{\boldsymbol{u}}_{\mathrm{N}}+\frac{\mu_{1}^{\text {vis }} \mu_{2}^{\text {vis }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}} \ddot{\boldsymbol{u}}_{\mathrm{N}} . \tag{6.2.6b}
\end{align*}
$$

Note that the above two ordinary differential equations have the same form as eq. (5.3.11). Namely, replacing $\sigma, \varepsilon, E, c_{1}$, and $c_{2}$ in eq. (5.3.11) by $\boldsymbol{f}_{\lambda}, \boldsymbol{u}_{\mathrm{N}}, \lambda^{\text {ela }}, \lambda_{1}^{\text {vis }}$, and $\lambda_{2}^{\text {vis }}$ yields eq. (6.2.6a). Similarly, replacing $\sigma, \varepsilon, E, c_{1}$, and $c_{2}$ in eq. (5.3.11) by $\boldsymbol{f}_{\mu}, \boldsymbol{u}_{\mathrm{N}}, \mu^{\text {ela }}, \mu_{1}^{\text {vis }}$, and $\mu_{2}^{\text {vis }}$ yields eq. (6.2.6b). Consequently, stress-strain relationship yields two ordinary differential equations with respect to $\boldsymbol{f}_{\lambda}$ and $\boldsymbol{f}_{\mu}$. Then, nodal force vector is given by $-J_{\lambda} \boldsymbol{f}_{\lambda}-J_{\mu} \boldsymbol{f}_{\mu}$.

### 6.3 Calculating inelastic deformation

Recall that dynamic deformation of an elastic body is formulated as eq. (4.3.2), yielding a canonical form of a set of ordinary differential equations:

$$
\begin{align*}
\dot{\boldsymbol{u}}_{\mathrm{N}} & =\boldsymbol{v}_{\mathrm{N}} \\
{\left[\begin{array}{cc}
M & -A \\
-A^{\top} &
\end{array}\right]\left[\begin{array}{c}
\dot{\boldsymbol{v}}_{\mathrm{N}} \\
\boldsymbol{\lambda}
\end{array}\right] } & =\left[\begin{array}{c}
-K \boldsymbol{u}_{\mathrm{N}}+\boldsymbol{f} \\
C\left(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}\right)
\end{array}\right] \tag{6.3.1}
\end{align*}
$$

where $C\left(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}\right)$ originates from equations for stabilizing constraints.
Replacing a set of elastic forces $-K \boldsymbol{u}_{\mathrm{N}}$ by a set of viscoelastic forces $-K \boldsymbol{u}_{\mathrm{N}}-B \dot{\boldsymbol{u}}_{\mathrm{N}}=$ $-K \boldsymbol{u}_{\mathrm{N}}-B \boldsymbol{v}_{\mathrm{N}}$, we have a canonical form to calculate dynamic deformation of a viscoelastic body:

$$
\begin{align*}
\dot{\boldsymbol{u}}_{\mathrm{N}} & =\boldsymbol{v}_{\mathrm{N}} \\
{\left[\begin{array}{cc}
M & -A \\
-A^{\top} &
\end{array}\right]\left[\begin{array}{c}
\dot{\boldsymbol{v}}_{\mathrm{N}} \\
\boldsymbol{\lambda}
\end{array}\right] } & =\left[\begin{array}{c}
-K \boldsymbol{u}_{\mathrm{N}}-B \boldsymbol{v}_{\mathrm{N}}+\boldsymbol{f} \\
C\left(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}\right)
\end{array}\right] \tag{6.3.2}
\end{align*}
$$

The state variables of this canonical form consists of $\boldsymbol{u}_{\mathrm{N}}$ and $\boldsymbol{v}_{\mathrm{N}}$.
Replacing a set of elastic forces $-K \boldsymbol{u}_{\mathrm{N}}$ by a set of rheological forces $-J_{\lambda} \boldsymbol{f}_{\lambda}-J_{\mu} \boldsymbol{f}_{\mu}$, and adding ordinary differential equations with respect to $\boldsymbol{f}_{\lambda}$ and $\boldsymbol{f}_{\mu}$, we have a canonical form to calculate dynamic deformation of a rheological body. In Maxwell model, adding eqs. (6.2.5a)(6.2.5b), a canonical form to calculate rheological deformation is described as

$$
\begin{align*}
\dot{\boldsymbol{u}}_{\mathrm{N}} & =\boldsymbol{v}_{\mathrm{N}} \\
{\left[\begin{array}{cc}
M & -A \\
-A^{\top} &
\end{array}\right]\left[\begin{array}{c}
\dot{\boldsymbol{v}}_{\mathrm{N}} \\
\boldsymbol{\lambda}
\end{array}\right] } & =\left[\begin{array}{c}
-J_{\lambda} \boldsymbol{f}_{\lambda}-J_{\mu} \boldsymbol{f}_{\mu}+\boldsymbol{f} \\
C\left(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}\right)
\end{array}\right] \\
\dot{\boldsymbol{f}}_{\lambda} & =-\frac{\lambda^{\mathrm{ela}}}{\lambda^{\text {vis }}} \boldsymbol{f}_{\lambda}+\lambda^{\mathrm{ela}} \boldsymbol{v}_{\mathrm{N}}  \tag{6.3.3}\\
\dot{\boldsymbol{f}}_{\mu} & =-\frac{\mu^{\mathrm{ela}}}{\mu^{\text {vis }}} \boldsymbol{f}_{\mu}+\mu^{\text {ela }} \boldsymbol{v}_{\mathrm{N}}
\end{align*}
$$

The state variables of this canonical form consists of $\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}, \boldsymbol{f}_{\lambda}$, and $\boldsymbol{f}_{\mu}$. Given $\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}, \boldsymbol{f}_{\lambda}$, and $\boldsymbol{f}_{\mu}$, this canonical form directly calculates their time-derivatives; $\dot{\boldsymbol{u}}_{\mathrm{N}}, \dot{\boldsymbol{v}}_{\mathrm{N}}, \dot{\boldsymbol{f}}_{\lambda}$, and $\dot{\boldsymbol{f}}_{\mu}$. In three-element model, adding eqs. (6.2.6a)(6.2.6b), a canonical form to calculate rheological deformation is described as

$$
\begin{align*}
& \dot{\boldsymbol{u}}_{\mathrm{N}}=\boldsymbol{v}_{\mathrm{N}} \\
& {\left[\begin{array}{cc}
M & -A \\
-A^{\top}
\end{array}\right]\left[\begin{array}{c}
\dot{\boldsymbol{v}}_{\mathrm{N}} \\
\boldsymbol{\lambda}
\end{array}\right]=\left[\begin{array}{c}
-J_{\lambda} \boldsymbol{f}_{\lambda}-J_{\mu} \boldsymbol{f}_{\mu}+\boldsymbol{f} \\
C\left(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}\right)
\end{array}\right]} \\
& \dot{\boldsymbol{f}}_{\lambda}=-\frac{\lambda^{\text {ela }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}} \boldsymbol{f}_{\lambda}+\frac{\lambda^{\text {ela }} \lambda_{2}^{\text {vis }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}} \boldsymbol{v}_{\mathrm{N}}+\frac{\lambda_{1}^{\text {vis }} \lambda_{2}^{\text {vis }}}{\lambda_{1}^{\text {vis }}+\lambda_{2}^{\text {vis }}} \dot{\boldsymbol{v}}_{\mathrm{N}}  \tag{6.3.4}\\
& \dot{\boldsymbol{f}}_{\mu}=-\frac{\mu^{\text {ela }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}} \boldsymbol{f}_{\mu}+\frac{\mu^{\text {ela }} \mu_{2}^{\text {vis }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}} \boldsymbol{v}_{\mathrm{N}}+\frac{\mu_{1}^{\text {vis }} \mu_{2}^{\text {vis }}}{\mu_{1}^{\text {vis }}+\mu_{2}^{\text {vis }}} \dot{\boldsymbol{v}}_{\mathrm{N}}
\end{align*}
$$

The state variables of this canonical form consists of $\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}, \boldsymbol{f}_{\lambda}$, and $\boldsymbol{f}_{\mu}$. Given $\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}, \boldsymbol{f}_{\lambda}$, and $\boldsymbol{f}_{\mu}$, the first and second equations calculate time-derivatives $\dot{\boldsymbol{u}}_{\mathrm{N}}$ and $\dot{\boldsymbol{v}}_{\mathrm{N}}$. Then, applying calculated $\dot{\boldsymbol{v}}_{\mathrm{N}}$ to the third and fourth equations, we can calculate time-derivatives $\dot{\boldsymbol{f}}_{\lambda}$, and $\dot{\boldsymbol{f}}_{\mu}$. Classes Triangle_ThreeElementModel and Body_ThreeElementModel were defined. The former is a subclass of Triangle and the latter is a subclass of Body. Instance of class Triangle_ThreeElementModel or Body_ThreeElementModel includes physical parameters specific to three-element model, $\lambda_{1}^{\text {vis }}, \mu_{1}^{\text {vis }}, \lambda_{2}^{\text {vis }}$, and $\mu_{2}^{\text {vis }}$.

Example Let us calculate the dynamic deformation of a two-dimensional rheological square body of width $w$ (Fig. 4.2). We apply three-element model to describe rheological deformation. Let us divide the square region into $3 \times 3 \times 2$ triangles. During time interval [ $0, t_{\text {push }}$ ], the bottom of the body is fixed to the floor and edge $\mathrm{P}_{14} \mathrm{P}_{15}$ moves downward at a constant velocity $v_{\text {push }}$. During [ $t_{\text {push }}, t_{\text {hold }}$ ], the bottom remains fixed and and edge $\mathrm{P}_{14} \mathrm{P}_{15}$ keeps its position. During [ $t_{\text {hold }}, t_{\text {end }}$ ], the bottom remains fixed while $\mathrm{P}_{14} \mathrm{P}_{15}$ is released. Figure


Figure 6.1: Dynamic deformation of a rheological square body ( $3 \times 3 \times 2$ triangles)
6.1 shows a snapshot of the computation result with $w=30 \mathrm{~cm}, h=1 \mathrm{~cm}, E=1.0 \mathrm{MPa}$, $c_{1}=40 \mathrm{~Pa} \cdot \mathrm{~s}, c_{2}=2.0 \mathrm{MPa} \cdot \mathrm{s}, \nu=0.48, \rho=1.0 \mathrm{~g} / \mathrm{cm}^{3}, t_{\text {push }}=1.0 \mathrm{~s}, t_{\text {hold }}=2.0 \mathrm{~s}$, and $v_{\text {push }}=16 \mathrm{~cm} / \mathrm{s}$. Deformation remains even after pushing motion is released. Figure 6.2 shows a snapshot of the computation result under a finer mesh; the square region consists of $9 \times 9 \times 2$ triangles, resulting a detailed description on deformation.

### 6.4 Inhomogeneous rheological deformation

Let us formulate rheological deformation of an inhomogeneous body. Assume that a twodimensional body consists of a finite number of subregions, over each of which deformation parameters are uniform. Let $\boldsymbol{u}_{\mathrm{N}}^{p}$ be a collective vector of nodal point displacements involved in subregion $\mathrm{S}_{p}$. Note that inelastic deformation of subregion $\mathrm{S}_{p}$ causes nodal point forces corresponding to $\boldsymbol{u}_{p}$. Let $\boldsymbol{f}_{\mathrm{N}}^{p}$ be a collective vector of the nodal point forces. Let $J_{\lambda}^{p}$ and $J_{\mu}^{p}$ be partial connection matrices of subregion $S_{p}$. Replacing nodal displacement vector $\boldsymbol{u}_{\mathrm{N}}$ in eq. (6.2.2) by $\boldsymbol{u}_{\mathrm{N}}^{p}$ as well as connection matrices $J_{\lambda}$ and $J_{\mu}$ by partial connection matrices $J_{\lambda}^{p}$ and $J_{\mu}^{p}$, we obtain

$$
\boldsymbol{f}_{\mathrm{N}}^{p}=-J_{\lambda}^{p} \int_{0}^{t} r_{\lambda}^{p}\left(t-t^{\prime}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}^{p}\left(t^{\prime}\right) \mathrm{d} t^{\prime}-J_{\mu}^{p} \int_{0}^{t} r_{\mu}^{p}\left(t-t^{\prime}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}^{p}\left(t^{\prime}\right) \mathrm{d} t^{\prime}
$$



Figure 6.2: Dynamic deformation of a rheological square body ( $9 \times 9 \times 2$ triangles $)$

Relaxation functions $r_{\lambda}^{p}\left(t-t^{\prime}\right)$ and $r_{\mu}^{p}\left(t-t^{\prime}\right)$ are defined for individual subregions. Introducing

$$
\begin{aligned}
\boldsymbol{f}_{\lambda}^{p}(t) & =\int_{0}^{t} r_{\lambda}^{p}\left(t-t^{\prime}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}^{p}\left(t^{\prime}\right) \mathrm{d} t^{\prime} \\
\boldsymbol{f}_{\mu}^{p}(t) & =\int_{0}^{t} r_{\mu}^{p}\left(t-t^{\prime}\right) \dot{\boldsymbol{u}}_{\mathrm{N}}^{p}\left(t^{\prime}\right) \mathrm{d} t^{\prime}
\end{aligned}
$$

a set of nodal rheological forces is simply described as

$$
\boldsymbol{f}_{\mathrm{N}}^{p}=-J_{\lambda}^{p} \boldsymbol{f}_{\lambda}^{p}(t)-J_{\mu}^{p} \boldsymbol{f}_{\mu}^{p}(t) .
$$

Synthesizing nodal point forces of all subregions, we obtain the total nodal force vector:

$$
\boldsymbol{f}_{\mathrm{N}}=\bigoplus_{p} \boldsymbol{f}_{\mathrm{N}}^{p}
$$

Note that the above calculation is similar to eq. (4.4.1), where three nodal forces corresponding to individual triangles are distributed and added for the total nodal force vector. In the above equation, three or more nodal forces corresponding to individual subregions are distributed and added for the total nodal force vector.

Let us demonstrate the above calculation process by a body consisting of two subregions (Fig. 6.3). Subregion $\mathrm{S}_{1}$ (darker color) consists of two triangles $\mathrm{T}_{1}=\triangle \mathrm{P}_{1} \mathrm{P}_{2} \mathrm{P}_{5}$ and $\mathrm{T}_{3}=$


Figure 6.3: Body consisting of two subregions
$\triangle \mathrm{P}_{5} \mathrm{P}_{4} \mathrm{P}_{2}$ while subregion $\mathrm{S}_{2}$ (lighter color) consists of two triangles $\mathrm{T}_{2}=\triangle \mathrm{P}_{2} \mathrm{P}_{3} \mathrm{P}_{5}$ and $\mathrm{T}_{4}=\triangle \mathrm{P}_{6} \mathrm{P}_{5} \mathrm{P}_{3}$. Thus, subregion $\mathrm{S}_{1}$ involves nodal point $\mathrm{P}_{1}, \mathrm{P}_{2}, \mathrm{P}_{4}$, and $\mathrm{P}_{5}$ while subregion $\mathrm{S}_{2}$ involves nodal point $\mathrm{P}_{2}, \mathrm{P}_{3}, \mathrm{P}_{5}$, and $\mathrm{P}_{6}$. Nodal force vectors of the two subregions are described as

$$
f_{\mathrm{N}}^{1}=\left[\begin{array}{l}
f_{1}^{1} \\
f_{1}^{1} \\
f_{1}^{1} \\
f_{5}^{2}
\end{array}\right], \quad f_{\mathrm{N}}^{2}=\left[\begin{array}{l}
f_{2}^{2} \\
f_{3}^{2} \\
f_{2}^{2} \\
f_{6}^{2}
\end{array}\right]
$$

The total nodal force vector is then given by

$$
\boldsymbol{f}_{\mathrm{N}}=\boldsymbol{f}_{\mathrm{N}}^{1} \oplus \boldsymbol{f}_{\mathrm{N}}^{2}=\left[\begin{array}{lll}
\boldsymbol{f}_{1}^{1} & & \\
\boldsymbol{f}_{2}^{1} & + & \boldsymbol{f}_{2}^{2} \\
\boldsymbol{f}_{1}^{1} & & \boldsymbol{f}_{3}^{2} \\
\boldsymbol{f}_{5}^{1} & + & \boldsymbol{f}_{5}^{2} \\
& & \boldsymbol{f}_{6}^{2}
\end{array}\right]
$$

The above equation provides a set of nodal point forces caused by the deformation of an inhomogeneous body. Noting that $(1,2,3,4)$ block rows and columns of partial connection matrix $J_{\lambda}^{1}$ correspond to $\mathrm{P}_{1}, \mathrm{P}_{2}, \mathrm{P}_{4}$, and $\mathrm{P}_{5}$, we find $(1,2,3) \times(1,2,3)$ blocks of $J_{\lambda}^{1,2,4}$ (triangle $\mathrm{T}_{1}$ ) contribute to $(1,2,3) \times(1,2,3)$ blocks of $J_{\lambda}^{1}$ and $(1,2,3) \times(1,2,3)$ blocks of $J_{\lambda}^{5,4,2}$ (triangle $\mathrm{T}_{3}$ ) contribute to $(4,3,2) \times(4,3,2)$ blocks of $J_{\lambda}^{1}$. Similarly, $(1,2,3,4)$ block rows and columns of partial connection matrix $J_{\lambda}^{2}$ correspond to $\mathrm{P}_{2}, \mathrm{P}_{3}, \mathrm{P}_{5}$, and $\mathrm{P}_{6}$, yielding that $(1,2,3) \times(1,2,3)$ blocks of $J_{\lambda}^{2,3,5}$ (triangle $\left.\mathrm{T}_{2}\right)$ contribute to $(1,2,3) \times(1,2,3)$ blocks of $J_{\lambda}^{2}$ and $(1,2,3) \times(1,2,3)$ blocks of $J_{\lambda}^{6,5,3}\left(\right.$ triangle $\left.\mathrm{T}_{4}\right)$ contribute to $(4,3,2) \times(4,3,2)$ blocks of $J_{\lambda}^{2}$. Based on these correspondences, we can calculate partial connection matrices of individual subregions.

Let us formulate rheological deformation under Maxwell model. Assume that the body consists of $n_{s}$ subregions. Deformation parameters $\lambda_{p}^{\text {ela }}, \lambda_{p}^{\text {vis }}, \mu_{p}^{\text {ela }}$, and $\mu_{p}^{\text {vis }}$ are uniform over subregion $\mathrm{S}_{p}$. Individual subregions may have different values of deformation parameters. A canonical form to calculate rheological deformation is then described as

$$
\begin{align*}
\dot{\boldsymbol{u}}_{\mathrm{N}} & =\boldsymbol{v}_{\mathrm{N}} \\
{\left[\begin{array}{cc}
M & -A \\
-A^{\top} &
\end{array}\right]\left[\begin{array}{c}
\dot{\boldsymbol{v}}_{\mathrm{N}} \\
\boldsymbol{\lambda}
\end{array}\right] } & =\left[\begin{array}{c}
\boldsymbol{f}_{\mathrm{N}}+\boldsymbol{f} \\
C\left(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}\right)
\end{array}\right] \\
\dot{\boldsymbol{f}}_{\lambda}^{p} & =-\frac{\lambda_{p}^{\text {ela }}}{\lambda_{p}^{\text {vis }}} \boldsymbol{f}_{\lambda}^{p}+\lambda_{p}^{\text {ela }} \boldsymbol{v}_{\mathrm{N}}^{p}, \quad\left(p=1,2, \cdots, n_{s}\right)  \tag{6.4.1}\\
\dot{\boldsymbol{f}}_{\mu}^{p} & =-\frac{\mu_{p}^{\text {ela }}}{\mu_{p}^{\text {vis }}} \boldsymbol{f}_{\mu}^{p}+\mu^{\text {ela }} \boldsymbol{v}_{\mathrm{N}}^{p}, \quad\left(p=1,2, \cdots, n_{s}\right)
\end{align*}
$$

The state variables of this canonical form consists of $\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}, \boldsymbol{f}_{\lambda}^{1}$ through $\boldsymbol{f}_{\lambda}^{n}$, and $\boldsymbol{f}_{\mu}^{1}$ through $\boldsymbol{f}_{\mu}^{n}$. Vectors $\boldsymbol{f}_{\lambda}^{p}$ and $\boldsymbol{f}_{\mu}^{p}$ correspond to subregion $\mathrm{S}_{p}$. Given $\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}, \boldsymbol{f}_{\lambda}^{1}$ through $\boldsymbol{f}_{\lambda}^{n}$, and
$\boldsymbol{f}_{\mu}^{1}$ through $\boldsymbol{f}_{\mu}^{n}$, this canonical form directly calculates their time-derivatives; $\dot{\boldsymbol{u}}_{\mathrm{N}}, \dot{\boldsymbol{v}}_{\mathrm{N}}, \dot{\boldsymbol{f}}_{\lambda}^{1}$ through $\dot{\boldsymbol{f}}_{\lambda}^{n}$, and $\dot{\boldsymbol{f}}_{\mu}^{1}$ through $\dot{\boldsymbol{f}}_{\mu}^{n}$.

Let us formulate rheological deformation under three-element model. Assume that the body consists of $n$ subregions. Deformation parameters $\lambda_{p}^{\text {ela }}, \lambda_{1, p}^{\text {vis }}, \lambda_{2, p}^{\text {vis }}, \mu_{p}^{\text {ela }}, \mu_{1, p}^{\text {vis }}$, and $\mu_{2, p}^{\text {vis }}$ are uniform over subregion $S_{p}$. Individual subregions may have different values of deformation parameters. A canonical form to calculate rheological deformation is then described as

$$
\begin{align*}
& \dot{\boldsymbol{u}}_{\mathrm{N}}=\boldsymbol{v}_{\mathrm{N}} \\
& {\left[\begin{array}{cc}
M & -A \\
-A^{\top} &
\end{array}\right]\left[\begin{array}{c}
\dot{\boldsymbol{v}}_{\mathrm{N}} \\
\boldsymbol{\lambda}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{f}_{\mathrm{N}}+\boldsymbol{f} \\
C\left(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}\right)
\end{array}\right]} \\
& \dot{\boldsymbol{f}}_{\lambda}^{p}=-\frac{\lambda_{p}^{\text {ela }}}{\lambda_{1, p}^{\text {vis }}+\lambda_{2, p}^{\text {vis }}} \boldsymbol{f}_{\lambda}^{p}+\frac{\lambda_{p}^{\text {ela }} \lambda_{2, p}^{\text {vis }}}{\lambda_{1, p}^{\text {vis }}+\lambda_{2, p}^{\text {vis }}} \boldsymbol{v}_{\mathrm{N}}^{p}+\frac{\lambda_{1, p}^{\text {vis }} \lambda_{2, p}^{\text {vis }}}{\lambda_{1, p}^{\text {vis }}+\lambda_{2, p}^{\text {vis }}} \dot{\boldsymbol{s}}_{\mathrm{N}}^{p}, \quad(p=1,2, \cdots, n)  \tag{6.4.2}\\
& \dot{\boldsymbol{f}}_{\mu}^{p}=-\frac{\mu_{p}^{\text {ela }}}{\mu_{1, p}^{\text {vis }}+\mu_{2, p}^{\text {vis }}} \boldsymbol{f}_{\mu}^{p}+\frac{\mu_{p}^{\text {ela }} \mu_{2, p}^{\text {vis }}}{\mu_{1, p}^{\text {vis }}+\mu_{2, p}^{\text {vis }}} \boldsymbol{v}_{\mathrm{N}}^{p}+\frac{\mu_{1, p}^{\text {vis }} \mu_{2, p}^{\text {vis }}}{\mu_{1, p}^{\text {vis }}+\mu_{2, p}^{\text {vis }}} \dot{v}_{\mathrm{N}}^{p}, \quad(p=1,2, \cdots, n)
\end{align*}
$$

The state variables of this canonical form consists of $\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}, \boldsymbol{f}_{\lambda}^{1}$ through $\boldsymbol{f}_{\lambda}^{n}$, and $\boldsymbol{f}_{\mu}^{1}$ through $\boldsymbol{f}_{\mu}^{n}$. Given $\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}, \boldsymbol{f}_{\lambda}^{1}$ through $\boldsymbol{f}_{\lambda}^{n}$, and $\boldsymbol{f}_{\mu}^{1}$ through $\boldsymbol{f}_{\mu}^{n}$, the first and second equations calculate time-derivatives $\dot{\boldsymbol{u}}_{\mathrm{N}}$ and $\dot{\boldsymbol{v}}_{\mathrm{N}}$. Then, applying calculated $\dot{\boldsymbol{v}}_{\mathrm{N}}$ to the third and fourth equations, we can calculate time-derivatives $\dot{\boldsymbol{f}}_{\lambda}^{1}$ through $\dot{\boldsymbol{f}}_{\lambda}^{n}$ and $\dot{\boldsymbol{f}}_{\mu}^{1}$ through $\dot{\boldsymbol{f}}_{\mu}^{n}$. Consequently, given state variables, we can calculate their time-derivatives, which provides a canonical form of a set of ordinary differential equations.

Example Let us calculate rheological deformation of a horizontally layered body. We apply three-element model. Class SubRegion_ThreeElementModel (a subclass of SubRegion) was introduced. Instance of this class includes physical parameters specific to three-element model, $\lambda_{1}^{\text {vis }}, \mu_{1}^{\text {vis }}, \lambda_{2}^{\text {vis }}$, and $\mu_{2}^{\text {vis }}$.

The layered body of width 10 cm and thickness $h=1 \mathrm{~cm}$ consists of two materials. Dark region (subregion 1) corresponds to a harder material specified by $E=1.0 \mathrm{MPa}, c_{1}=40 \mathrm{~Pa} \cdot \mathrm{~s}$, $c_{2}=2.0 \mathrm{MPa} \cdot \mathrm{s}, \nu=0.48$, and $\rho=1.0 \mathrm{~g} / \mathrm{cm}^{3}$ while light region (subregion 2 ) corresponds to a softer material specified by $E=0.2 \mathrm{MPa}, c_{1}=40 \mathrm{~Pa} \cdot \mathrm{~s}, c_{2}=2.0 \mathrm{MPa} \cdot \mathrm{s}, \nu=0.48$, and $\rho=1.0 \mathrm{~g} / \mathrm{cm}^{3}$. Namely, material in dark region is five-times harder than material in light region. Subregion 1 consists of $4 \times 18$ triangles and includes 60 nodal points, implying that $\boldsymbol{f}_{\lambda}^{1}$ and $\boldsymbol{f}_{\mu}^{1}$ are 60 dimensional vectors. Subregion 2 consists of $5 \times 18$ triangles and includes 60 nodal points, implying that $\boldsymbol{f}_{\lambda}^{2}$ and $\boldsymbol{f}_{\mu}^{2}$ are 60 dimensional vectors. State variable vector in the canonical form is thus described as

$$
\boldsymbol{q}=\left[\begin{array}{c}
\boldsymbol{u}_{\mathrm{N}} \\
\boldsymbol{v}_{\mathrm{N}} \\
\boldsymbol{f}_{\lambda}^{1} \\
\boldsymbol{f}_{\lambda}^{2} \\
\boldsymbol{f}_{\mu}^{1} \\
\boldsymbol{f}_{\mu}^{2}
\end{array}\right]
$$

Vector $\boldsymbol{q}$ consists of 640 elements.
During time interval [ $0, t_{\text {push }}$ ], the bottom of the body is fixed to the floor and the middle of the top surface moves downward at a constant velocity $v_{\text {push }}$. During [ $t_{\text {push }}, t_{\text {hold }}$ ], the bottom remains fixed and and the middle of the top surface keeps its position. During [ $t_{\text {hold }}$, $t_{\text {end }}$ ], the bottom remains fixed while the middle of the top surface is released. Here we apply $t_{\text {push }}=1.0 \mathrm{~s}, t_{\text {hold }}=2.0 \mathrm{~s}$, and $v_{\text {push }}=16 \mathrm{~cm} / \mathrm{s}$. We find that the softer layer (light region) deforms more than the harder layers (dark region). This deformation of a layered rheological


Figure 6.4: Dynamic deformation of a horizontally layered rheological body ( $9 \times 9 \times 2$ triangles)
body is different from deformation of a uniform rheological body (Fig. 6.2). The softer layer expands outward more in this deformation.

## Problems

1. Assuming that deformation of body material is characterized by parallel elastic model (Section 5.4.2), formulate the nodal force vector of the body.
