Chapter 7

Isotropy and Invariants

7.1 Isotropy

Isotropy in two-dimensional deformation *Isotropy* implies that a material shows the same deformation property at any direction. We formulate isotropic elasticity in two-dimensional deformation.

First, we formulate strain components along with an arbitrary direction. Let O–xy and O–$x'y'$ be coordinate systems fixed to two-dimensional space. Let α be angle between x-axis and x'-axis. Then, relationship between x, y coordinates and x', y' is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_\alpha & -S_\alpha \\ S_\alpha & C_\alpha \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

where $C_\alpha = \cos\alpha$ and $S_\alpha = \sin\alpha$. Equivalently,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} C_\alpha & S_\alpha \\ -S_\alpha & C_\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $\boldsymbol{u}$ be a two-dimensional displacement vector of an arbitrary point. Let $[\,u,\, v\,]^\top$ be its components at O–xy while $[\,u',\, v'\,]^\top$ be its components at O–$x'y'$. Then, we have the following relationship:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} C_\alpha & -S_\alpha \\ S_\alpha & C_\alpha \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}$$

Let $\boldsymbol{\varepsilon} = [\,\varepsilon_{xx},\, \varepsilon_{yy},\, 2\varepsilon_{xy}\,]^\top$ be strain vector at O–xy while $\boldsymbol{\varepsilon}' = [\,\varepsilon_{x'x'},\, \varepsilon_{y'y'},\, 2\varepsilon_{x'y'}\,]^\top$ be strain vector at O–$x'y'$. Namely,

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x}, & \varepsilon_{yy} &= \frac{\partial v}{\partial y}, & 2\varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \varepsilon_{x'x'} &= \frac{\partial u'}{\partial x'}, & \varepsilon_{y'y'} &= \frac{\partial v'}{\partial y'}, & 2\varepsilon_{x'y'} &= \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} \end{aligned}$$

Note that deformation can be formulated either in O–xy or O–$x'y'$.

Next, let us derive the relationship between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$. Noting that x depends on x', y'

with $\partial x'/\partial x = C_\alpha$ and $\partial y'/\partial x = -S_\alpha$, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} \\ &= \frac{\partial(C_\alpha u' - S_\alpha v')}{\partial x'}C_\alpha + \frac{\partial(C_\alpha u' - S_\alpha v')}{\partial y'}(-S_\alpha) \\ &= C_\alpha^2\frac{\partial u'}{\partial x'} + S_\alpha^2\frac{\partial v'}{\partial y'} + (-C_\alpha S_\alpha)\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\end{aligned}$$

which directly yields

$$\varepsilon_{xx} = C_\alpha^2\varepsilon_{x'x'} + S_\alpha^2\varepsilon_{y'y'} + (-C_\alpha S_\alpha)\cdot 2\varepsilon_{x'y'} \tag{7.1.1}$$

Similarly,

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{\partial v}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial v}{\partial y'}\frac{\partial y'}{\partial y} \\ &= \frac{\partial(S_\alpha u' + C_\alpha v')}{\partial x'}S_\alpha + \frac{\partial(S_\alpha u' + C_\alpha v')}{\partial y'}C_\alpha \\ &= S_\alpha^2\frac{\partial u'}{\partial x'} + C_\alpha^2\frac{\partial v'}{\partial y'} + C_\alpha S_\alpha\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\end{aligned}$$

which directly yields

$$\varepsilon_{yy} = S_\alpha^2\varepsilon_{x'x'} + C_\alpha^2\varepsilon_{y'y'} + C_\alpha S_\alpha\cdot 2\varepsilon_{x'y'} \tag{7.1.2}$$

Also, we have

$$\begin{aligned}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial v}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial v}{\partial y'}\frac{\partial y'}{\partial x} \\ &= 2C_\alpha S_\alpha\frac{\partial u'}{\partial x'} + (-2C_\alpha S_\alpha)\frac{\partial v'}{\partial y'} + (C_\alpha^2 - S_\alpha^2)\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\end{aligned}$$

which directly yields

$$2\varepsilon_{xy} = 2C_\alpha S_\alpha\varepsilon_{x'x'} + (-2C_\alpha S_\alpha)\varepsilon_{y'y'} + (C_\alpha^2 - S_\alpha^2)\cdot 2\varepsilon_{x'y'} \tag{7.1.3}$$

Consequently, we have

$$\boldsymbol{\varepsilon} = P(\alpha)\,\boldsymbol{\varepsilon}' \tag{7.1.4}$$

where

$$P(\alpha) = \begin{bmatrix} C_\alpha^2 & S_\alpha^2 & -C_\alpha S_\alpha \\ S_\alpha^2 & C_\alpha^2 & C_\alpha S_\alpha \\ 2C_\alpha S_\alpha & -2C_\alpha S_\alpha & C_\alpha^2 - S_\alpha^2 \end{bmatrix} \tag{7.1.5}$$

Note that matrix $P(\alpha)$ characterizes strain components along with an arbitrary direction.

Let us formulate linear isotropic elasticity. Let D be elasticity matrix at O–xy. Then, strain energy density is formulated as $(1/2)\boldsymbol{\varepsilon}^\top D\boldsymbol{\varepsilon}$. This strain energy density can be rewritten as

$$\frac{1}{2}\boldsymbol{\varepsilon}^\top D\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\varepsilon}')^\top\,P(\alpha)^\top\,D\,P(\alpha)\,(\boldsymbol{\varepsilon}')$$

implying that elasticity matrix at O–$x'y'$ is described as $P(\alpha)^\top D P(\alpha)$. Isotropic elasticity requires a condition that elasticity matrix is invariant against angle α. Namely,

$$P(\alpha)^\top \, D \, P(\alpha) = D \tag{7.1.6}$$

should be satisfied for an arbitrary α for isotropic elasticity. Let us describe symmetric matrix D as

$$D = \begin{bmatrix} D_{xx} & D_{xy} & D_{x\gamma} \\ D_{xy} & D_{yy} & D_{y\gamma} \\ D_{x\gamma} & D_{y\gamma} & D_{\gamma\gamma} \end{bmatrix}$$

Isotropic elasticity requires $P(\pi/2)^\top \, D \, P(\pi/2) = D$. Noting that

$$\begin{aligned} P(\pi/2)^\top \, D \, P(\pi/2) &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_{xx} & D_{xy} & D_{x\gamma} \\ D_{xy} & D_{yy} & D_{y\gamma} \\ D_{x\gamma} & D_{y\gamma} & D_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} D_{yy} & D_{xy} & D_{y\gamma} \\ D_{xy} & D_{xx} & D_{x\gamma} \\ D_{y\gamma} & D_{x\gamma} & D_{\gamma\gamma} \end{bmatrix} \end{aligned}$$

we find $D_{xx} = D_{yy}$ and $D_{x\gamma} = D_{y\gamma}$. Then, elasticity matrix D can be described as

$$D = \begin{bmatrix} D_{xx} & D_{xy} & D_{x\gamma} \\ D_{xy} & D_{xx} & D_{x\gamma} \\ D_{x\gamma} & D_{x\gamma} & D_{\gamma\gamma} \end{bmatrix}$$

Isotropic elasticity requires $P(\pi/4)^\top \, D \, P(\pi/4) = D$. Noting that

$$P(\pi/4)^\top \, D \, P(\pi/4) =$$
$$\begin{bmatrix} D_{xx}/2 + D_{xy}/2 + 2D_{x\gamma} + D_{\gamma\gamma} & D_{xx}/2 + D_{xy}/2 - D_{\gamma\gamma} & 0 \\ D_{xx}/2 + D_{xy}/2 - D_{\gamma\gamma} & D_{xx}/2 + D_{xy}/2 - 2D_{x\gamma} + D_{\gamma\gamma} & 0 \\ 0 & 0 & D_{xx}/2 - D_{xy}/2 \end{bmatrix}$$

we find $D_{x\gamma} = 0$ and $D_{xx} - D_{xy} - 2D_{\gamma\gamma} = 0$. Letting $\lambda = D_{xy}$ and $\mu = D_{\gamma\gamma}$, elasticity matrix D can be described as

$$D = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{7.1.7}$$

The above equation can be described as

$$D = \lambda I_\lambda + \mu I_\mu \tag{7.1.8}$$

where

$$I_\lambda = \begin{bmatrix} 1 & 1 & \\ 1 & 1 & \\ & & \end{bmatrix}, \quad I_\mu = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix} \tag{7.1.9}$$

The above matrices satisfy

$$P(\alpha)^\top I_\lambda P(\alpha) = I_\lambda, \quad P(\alpha)^\top I_\mu P(\alpha) = I_\mu \tag{7.1.10}$$

for an arbitrary α. Consequently, we find that elasticity matrix D given in eq. (7.1.7) satisfies isotropy condition eq. (7.1.6).

Material-specific constants λ and μ are referred to as *Lamé's constants*. Lamé's constants are described by Young's modulus E and Poisson's ratio ν as follows:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{7.1.11}$$

Additionally, constant μ is equal to shear elasticity modulus.

Let $\boldsymbol{E} = [\,E_{xx},\, E_{yy},\, 2E_{xy}\,]^\top$ be Green strain with respect to O–xy and $\boldsymbol{E}' = [\,E_{x'x'},\, E_{y'y'},\, 2E_{x'y'}\,]^\top$ be Green strain with respect to O–$x'y'$. Noting that

$$\begin{aligned}
u_x &= C_\alpha^2 u'_{x'} + (-C_\alpha S_\alpha) u'_{y'} + (-C_\alpha S_\alpha) v'_{x'} + S_\alpha^2 v'_{y'} \\
u_y &= C_\alpha S_\alpha u'_{x'} + C_\alpha^2 u'_{y'} + (-S_\alpha^2) v'_{x'} + (-C_\alpha S_\alpha) v'_{y'} \\
v_x &= C_\alpha S_\alpha u'_{x'} + (-S_\alpha^2) u'_{y'} + C_\alpha^2 v'_{x'} + (-C_\alpha S_\alpha) v'_{y'} \\
v_y &= S_\alpha^2 u'_{x'} + C_\alpha S_\alpha u'_{y'} + C_\alpha S_\alpha v'_{x'} + C_\alpha^2 v'_{y'}
\end{aligned}$$

we find the following equations:

$$E_{xx} = C_\alpha^2 E_{x'x'} + S_\alpha^2 E_{y'y'} + (-C_\alpha S_\alpha)\cdot 2E_{x'y'} \tag{7.1.12a}$$

$$E_{yy} = S_\alpha^2 E_{x'x'} + C_\alpha^2 E_{y'y'} + C_\alpha S_\alpha \cdot 2E_{x'y'} \tag{7.1.12b}$$

$$2E_{xy} = 2C_\alpha S_\alpha E_{x'x'} + (-2C_\alpha S_\alpha) E_{y'y'} + (C_\alpha^2 - S_\alpha^2)\cdot 2E_{x'y'} \tag{7.1.12c}$$

The above equations agree with eqs. (7.1.1)(7.1.2)(7.1.3), implying that relationship between $\boldsymbol{E}$ and $\boldsymbol{E}'$ is formulated by eqs. (7.1.4)(7.1.5). Consequently, elasticity matrix D can be described as $D = \lambda I_\lambda + \mu I_\mu$ even when we apply Green strain $\boldsymbol{E}$ instead of Cauchy strain $\boldsymbol{\varepsilon}$. Namely, strain energy density is formulated as follows:

$$\frac{1}{2}\boldsymbol{E}^\top (\lambda I_\lambda + \mu I_\mu)\boldsymbol{E} \tag{7.1.13}$$

when material exhibits linear isotropic elasticity.

Isotropy in three-dimensional deformation Let O–xyz and O–$x'y'z'$ be coordinate systems fixed to three-dimensional space. Assume that relationship between $\boldsymbol{x} = [\,x,\, y,\, z\,]^\top$ and $\boldsymbol{x}' = [\,x',\, y',\, z'\,]^\top$ is described as $\boldsymbol{x} = R\boldsymbol{x}'$, where

$$R = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

is an orthogonal matrix. Equivalently, we have $\boldsymbol{x}' = R^\top \boldsymbol{x}$, which directly yields

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let $\boldsymbol{u}$ be a three-dimensional displacement vector of an arbitrary point. Let $[\,u,\, v\,]^\top w$ be its components at O–xyz while $[\,u',\, v',\, w'\,]^\top$ be its components at O–$x'y'z'$. Then, we have the following relationship:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}$$

Let $\boldsymbol{\varepsilon}$ be strain vector at O–xyz while $\boldsymbol{\varepsilon}'$ be strain vector at O–$x'y'z'$. Strain components then satisfy the following equations:

$$\begin{aligned}
\varepsilon_{xx} &= a_x^2\varepsilon_{x'x'} + b_x^2\varepsilon_{y'y'} + c_x^2\varepsilon_{z'z'} + b_xc_x\cdot 2\varepsilon_{y'z'} + c_xa_x\cdot 2\varepsilon_{z'x'} + a_xb_x\cdot 2\varepsilon_{x'y'}\\
\varepsilon_{yy} &= a_y^2\varepsilon_{x'x'} + b_y^2\varepsilon_{y'y'} + c_y^2\varepsilon_{z'z'} + b_yc_y\cdot 2\varepsilon_{y'z'} + c_ya_y\cdot 2\varepsilon_{z'x'} + a_yb_y\cdot 2\varepsilon_{x'y'}\\
\varepsilon_{zz} &= a_z^2\varepsilon_{x'x'} + b_z^2\varepsilon_{y'y'} + c_z^2\varepsilon_{z'z'} + b_zc_z\cdot 2\varepsilon_{y'z'} + c_za_z\cdot 2\varepsilon_{z'x'} + a_zb_z\cdot 2\varepsilon_{x'y'}\\
2\varepsilon_{yz} &= 2a_ya_z\varepsilon_{x'x'} + 2b_yb_z\varepsilon_{y'y'} + 2c_yz_z\varepsilon_{z'z'}\\
&\quad + (b_yc_z + b_zc_y)\cdot 2\varepsilon_{y'z'} + (c_ya_z + c_za_y)\cdot 2\varepsilon_{z'x'} + (a_yb_z + a_zb_y)\cdot 2\varepsilon_{x'y'}\\
2\varepsilon_{zx} &= 2a_za_x\varepsilon_{x'x'} + 2b_zb_x\varepsilon_{y'y'} + 2c_zz_x\varepsilon_{z'z'}\\
&\quad + (b_zc_x + b_xc_z)\cdot 2\varepsilon_{y'z'} + (c_za_x + c_xa_z)\cdot 2\varepsilon_{z'x'} + (a_zb_x + a_xb_z)\cdot 2\varepsilon_{x'y'}\\
2\varepsilon_{xy} &= 2a_xa_y\varepsilon_{x'x'} + 2b_xb_y\varepsilon_{y'y'} + 2c_xz_y\varepsilon_{z'z'}\\
&\quad + (b_xc_y + b_yc_x)\cdot 2\varepsilon_{y'z'} + (c_xa_y + c_ya_x)\cdot 2\varepsilon_{z'x'} + (a_xb_y + a_yb_x)\cdot 2\varepsilon_{x'y'}
\end{aligned}$$

Consequently, we have the following mapping:

$$\boldsymbol{\sigma} = P(R)\,\boldsymbol{\sigma}' \tag{7.1.14}$$

where

$$P(R) = \begin{bmatrix}
a_x^2 & b_x^2 & c_x^2 & b_xc_x & c_xa_x & a_xb_x\\
a_y^2 & b_y^2 & c_y^2 & b_yc_y & c_ya_y & a_yb_y\\
a_z^2 & b_z^2 & c_z^2 & b_zc_z & c_za_z & a_zb_z\\
2a_ya_z & 2b_yb_z & 2c_yz_z & b_yc_z + b_zc_y & c_ya_z + c_za_y & a_yb_z + a_zb_y\\
2a_za_x & 2b_zb_x & 2c_zz_x & b_zc_x + b_xc_z & c_za_x + c_xa_z & a_zb_x + a_xb_z\\
2a_xa_y & 2b_xb_y & 2c_xz_y & b_xc_y + b_yc_x & c_xa_y + c_ya_x & a_xb_y + a_yb_x
\end{bmatrix} \tag{7.1.15}$$

Note that matrix $P(R)$, which depends on orthogonal matrix R, determines strain components along with an arbitrary direction in three-dimensional space.

Assume that the material exhibits linear isotropic elasticity. Elastic matrix D then must satisfy

$$P(R)^\top D\,P(R) = D \tag{7.1.16}$$

for linear isotropic elasticity. Elasticity matrix D should be invariant with respect to the exchange between y- and z-axes, z- and x-axes, and x- and y-axes. Thus, eq. (7.1.16) must be satisfied for the mappings corresponding to the following permutation matrices:

$$T_{23} = \begin{bmatrix} 1 & & \\ & & 1\\ & 1 & \end{bmatrix},\quad T_{31} = \begin{bmatrix} & & 1\\ & 1 & \\ 1 & & \end{bmatrix},\quad T_{12} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$$

Solving $P(T_{23})^\top D\,P(T_{23}) = D$, $P(T_{31})^\top D\,P(T_{31}) = D$, and $P(T_{12})^\top D\,P(T_{12}) = D$, we find that the elasticity matrix can be described as

$$D = \left[\begin{array}{ccc|ccc}
D_{xx} & D_{xy} & D_{xy} & D_{x\alpha} & D_{x\beta} & D_{x\beta}\\
D_{xy} & D_{xx} & D_{xy} & D_{x\beta} & D_{x\alpha} & D_{x\beta}\\
D_{xy} & D_{xy} & D_{xx} & D_{x\beta} & D_{x\beta} & D_{x\alpha}\\ \hline
D_{x\alpha} & D_{x\beta} & D_{x\beta} & D_{\alpha\alpha} & D_{\alpha\beta} & D_{\alpha\beta}\\
D_{x\beta} & D_{x\alpha} & D_{x\beta} & D_{\alpha\beta} & D_{\alpha\alpha} & D_{\alpha\beta}\\
D_{x\beta} & D_{x\beta} & D_{x\alpha} & D_{\alpha\beta} & D_{\alpha\beta} & D_{\alpha\alpha}
\end{array}\right] \tag{7.1.17}$$

where D_{xx}, D_{xy}, $D_{\alpha\alpha}$, $D_{\alpha\beta}$, $D_{x\alpha}$, $D_{x\beta}$ are constant parameters. Elasticity matrix D should be invariant with respect to the rotation around z-axis by angle $\pi/4$. This rotation is given by

$$R_z(\pi/4) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & \\ 1/\sqrt{2} & 1/\sqrt{2} & \\ & & 1 \end{bmatrix}$$

Solving $P(R_z(\pi/4))^\top D\, P(R_z(\pi/4)) = D$, we have $D_{xx} = D_{xy} + 2D_{\alpha\alpha}$, $D_{\alpha\beta} = 0$, $D_{x\alpha} = 0$, and $D_{x\beta} = 0$. Letting $\lambda = D_{xy}$ and $\mu = D_{\alpha\alpha}$, the elasticity matrix can be described as

$$D = \left[\begin{array}{ccc|ccc} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ \hline & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{array}\right] \tag{7.1.18}$$

The above equation can be described as

$$D = \lambda I_\lambda + \mu I_\mu \tag{7.1.19}$$

where

$$I_\lambda = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array}\right], \quad I_\mu = \left[\begin{array}{ccc|ccc} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right] \tag{7.1.20}$$

The above matrices satisfy

$$P(R)^\top I_\lambda P(R) = I_\lambda, \quad P(R)^\top I_\mu P(R) = I_\mu \tag{7.1.21}$$

for an arbitrary orthogonal matrix R. Consequently, we find that elasticity matrix D given in eq. (7.1.18) satisfies isotropy condition eq. (7.1.16).

7.2 Invariants

Linear isotropy is characterized by matrices I_λ and I_μ. Let us investigate quantities invariant against rotation. Here we describe two-dimensional strain in a matrix form:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} \tag{7.2.1}$$

which is referred to as a *strain tensor*. Note that Jacobian of displacement vector $\boldsymbol{u}$ with respect $\boldsymbol{x}$ yields partial derivatives:

$$C = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}^\top} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

implying that strain tensor is described as $\varepsilon = (1/2)(C^\top + C)$. Characteristic equation of matrix ε is given as

$$\mid \lambda I - \varepsilon \mid = \lambda^2 - (\varepsilon_{xx} + \varepsilon_{yy})\lambda + (\varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2)$$

Applying eqs. (7.1.1)(7.1.2)(7.1.3) into the coefficients of the above equation, we find

$$\begin{aligned} \varepsilon_{xx} + \varepsilon_{yy} &= \varepsilon_{x'x'} + \varepsilon_{y'y'} \\ \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2 &= \varepsilon_{x'x'}\varepsilon_{y'y'} - \varepsilon_{x'y'}^2 \end{aligned}$$

Namely,

$$I_1 = \operatorname{tr} \varepsilon = \varepsilon_{xx} + \varepsilon_{yy} \tag{7.2.2}$$
$$I_2 = \det \varepsilon = \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2 \tag{7.2.3}$$

are invariant with respect to rotation, suggesting that any function of I_1 and I_2 is invariant against rotation. Let $W(I_1, I_2)$ be a strain energy density. Strain-stress relationship derived from $W(I_1, I_2)$ exhibits isotropic elasticity. Letting $W_1 = \partial W/\partial I_1$ and $W_2 = \partial W/\partial I_2$, we have the strain-stress relationship as

$$\begin{aligned} \sigma_{xx} &= \frac{\partial W}{\partial \varepsilon_{xx}} = W_1(I_1, I_2) + W_2(I_1, I_2)\, \varepsilon_{yy} \\ \sigma_{yy} &= \frac{\partial W}{\partial \varepsilon_{yy}} = W_1(I_1, I_2) + W_2(I_1, I_2)\, \varepsilon_{xx} \\ \sigma_{xy} &= \frac{\partial W}{\partial (2\varepsilon_{xy})} = -\frac{1}{2} W_2(I_1, I_2) \cdot 2\varepsilon_{xy} \end{aligned}$$

Consequently, strain energy density given by $W(I_1, I_2)$ characterizes isotropic elasticity.

The above discussion is applied to linear isotropic elasticity. Noting that $\boldsymbol{\varepsilon}^\top I_\lambda \boldsymbol{\varepsilon} = I_1^2$ and $\boldsymbol{\varepsilon}^\top I_\mu \boldsymbol{\varepsilon} = 2I_1^2 - 4I_2$, strain energy density of a linear isotropic material under two-dimensional deformation is described as follows:

$$W(I_1, I_2) = \frac{1}{2}\left\{\lambda I_1^2 + \mu(2I_1^2 - 4I_2)\right\}$$

Its partial derivatives are given as

$$W_1(I_1, I_2) = (\lambda + 2\mu)I_1, \qquad W_2(I_1, I_2) = -2\mu$$

which yield the following stress-strain relationship:

$$\begin{aligned} \sigma_{xx} &= (\lambda + 2\mu)I_1 + (-2\mu)\varepsilon_{yy} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} \\ \sigma_{yy} &= (\lambda + 2\mu)I_1 + (-2\mu)\varepsilon_{xx} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} \\ \sigma_{xy} &= -\frac{1}{2}(-2\mu) \cdot 2\varepsilon_{xy} = \mu(2\varepsilon_{xy}) \end{aligned}$$

that is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \\ \lambda & \lambda + 2\mu & \\ & & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix}$$

The above equations directly result in eq. (7.1.7).

Let us describe three-dimensional strain in a matrix form:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{7.2.4}$$

Letting $C = \partial \boldsymbol{u} / \partial \boldsymbol{x}^\top$ be Jacobian of displacement vector $\boldsymbol{u}$ with respect $\boldsymbol{x}$, we find that strain tensor is described as $\varepsilon = (1/2)(C^\top + C)$. Coefficients of characteristic equation of matrix ε determine invariants:

$$I_1 = \operatorname{tr} \varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{7.2.5}$$

$$I_2 = \bar{\operatorname{tr}}\, \varepsilon = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{vmatrix} + \begin{vmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{yz} & \varepsilon_{zz} \end{vmatrix} + \begin{vmatrix} \varepsilon_{zz} & \varepsilon_{zx} \\ \varepsilon_{zx} & \varepsilon_{xx} \end{vmatrix} \tag{7.2.6}$$

$$I_3 = \det \varepsilon = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{vmatrix} \tag{7.2.7}$$

Let $W(I_1, I_2, I_3)$ be a strain energy density. Strain-stress relationship derived from $W(I_1, I_2, I_3)$ exhibits isotropic elasticity.

The above discussion is applied to linear isotropic elasticity. Noting that $\boldsymbol{\varepsilon}^\top I_\lambda \boldsymbol{\varepsilon} = I_1^2$ and $\boldsymbol{\varepsilon}^\top I_\mu \boldsymbol{\varepsilon} = 2I_1^2 - 4I_2$, strain energy density of a linear isotropic material under three-dimensional deformation is described as follows:

$$W(I_1, I_2) = \frac{1}{2} \left\{ \lambda I_1^2 + \mu (2I_1^2 - 4I_2) \right\}$$

The above equation results in eq. (7.1.18).

Problems

1. Show eq. (7.1.10).
2. Show eqs. (7.1.12a)(7.1.12b)(7.1.12c).
3. Show eq. (7.1.17).
4. Show eq. (7.1.18).