Mechanics of Soft Bodies

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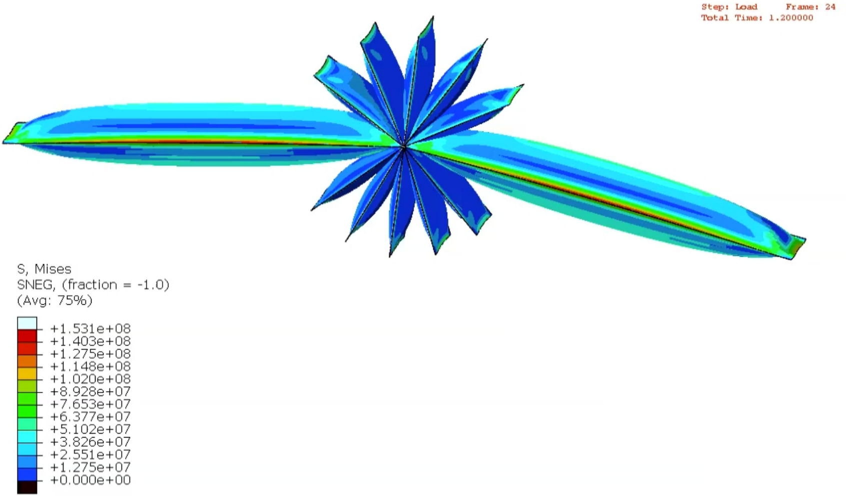
Agenda

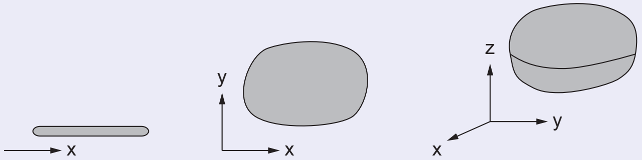
Finite Element Method (FEM)

inflatable link simulation



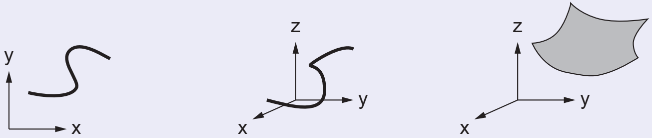
Soft Body Models

Soft-material Robots



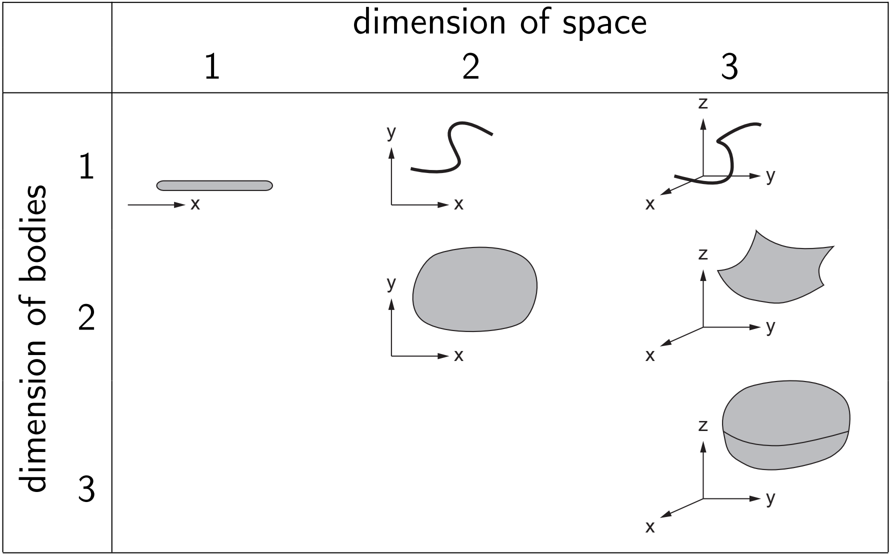
1D model 2D model 3D model

Geometrically Deformable Robots



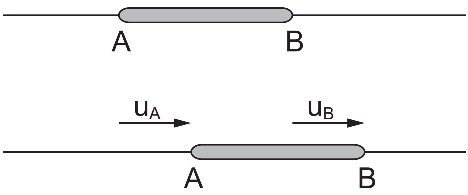
linear in 2D linear in 3D planar in 3D

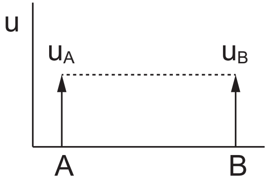
Soft Body Models



One-dimensional Soft Body Model

one-dimensional soft robot AB acts as

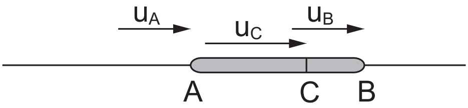




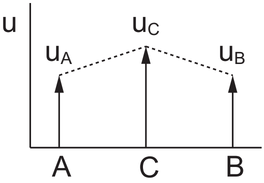
displacements

Can we conclude that AB moves but does not deform?

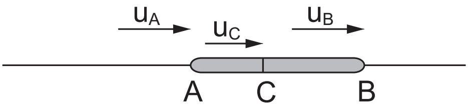
One-dimensional Soft Body Model



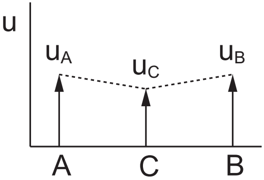
left half expands



displacements

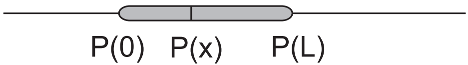


left half shrinks

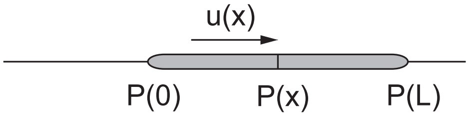


displacements

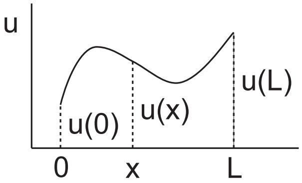
One-dimensional Soft Body Model



natural state



moved and deformed state

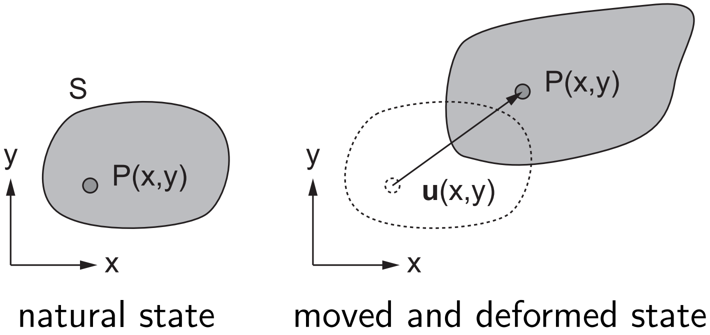


displacement function

the motion and deformation: specified by function $u(x)$, where $x \in [\,0,\; L\,]$

Two-dimensional Soft Body Model

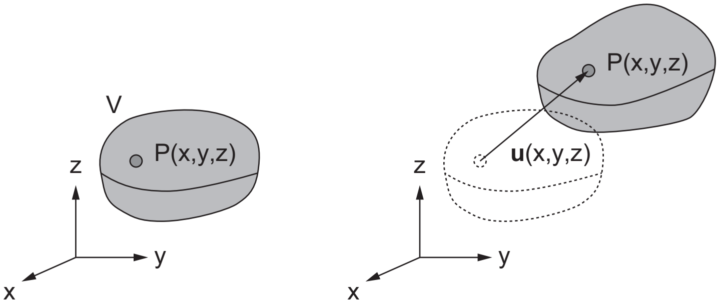
two-dimensional soft robot S acts as



The motion and deformation: specified by a vector function $\boldsymbol{u}(x, y)$, that is, by its two components $u(x, y)$ and $v(x, y)$

Three-dimensional Soft Body Model

three-dimensional soft robot V acts as



natural state　　moved and deformed state

The motion and deformation: specified by a vector function $\boldsymbol{u}(x, y, z)$, that is, by its three components $u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$

Approach

Energies

motion	kinetic energy T
deformation	strain potential energy U
	strain and stress

Calculation

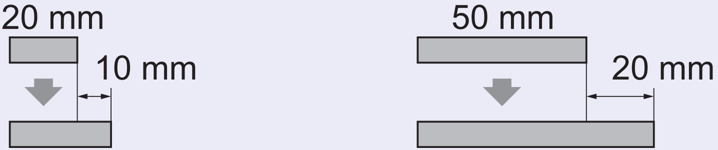
finite element approximation

divide-and-conquer approach

piecewise linear approximation

Strain and Stress

Which deforms more?



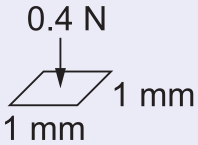
Strain

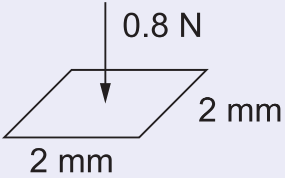
$$\text{strain} = \frac{\text{deformation}}{\text{size}}$$

$$\varepsilon = \frac{10\,\text{mm}}{20\,\text{mm}} = 0.50 \qquad \varepsilon = \frac{20\,\text{mm}}{50\,\text{mm}} = 0.40$$

Strain and Stress

Which pushes stronger?





Stress

$$\text{stress} = \frac{\text{force}}{\text{area}}$$

$$\sigma = \frac{0.4\,\mathrm{N}}{(1\,\mathrm{mm})^2} = 0.40\,\mathrm{MPa} \qquad \sigma = \frac{0.8\,\mathrm{N}}{(2\,\mathrm{mm})^2} = 0.20\,\mathrm{MPa}$$

Strain and Stress (Units)

Strain

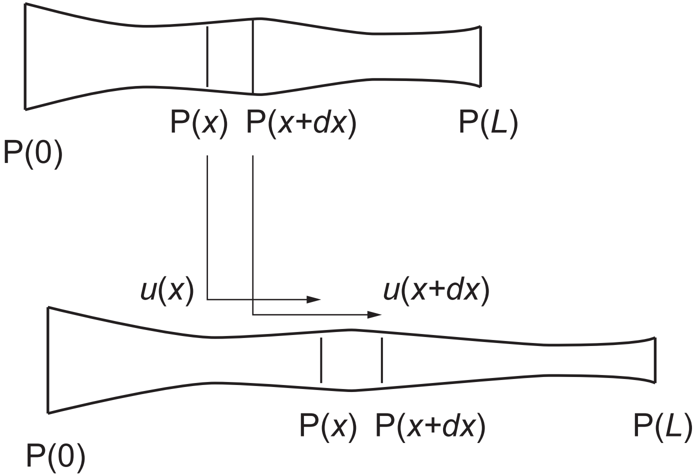
$$\frac{\text{deformation}}{\text{size}} = \frac{\mathrm{m}}{\mathrm{m}} = 1$$

Stress

$$\frac{\text{force}}{\text{area}} = \frac{\mathrm{N}}{\mathrm{m}^2} = \ \mathrm{Pa}$$

$$\frac{\mathrm{N}}{\mathrm{mm}^2} = \frac{\mathrm{N}}{(10^{-3}\,\mathrm{m})^2} = \frac{\mathrm{N}}{10^{-6}\,\mathrm{m}^2} = 10^6 \frac{\mathrm{N}}{\mathrm{m}^2} = 10^6\,\mathrm{Pa} = \ \mathrm{MPa}$$

One-dimensional Deformation



strain of small region $\mathrm{P}(x)\mathrm{P}(x + \mathrm{d}x)$

One-dimensional Deformation

$$
\begin{aligned}
\text{extension} &= u(x + \mathrm{d}x) - u(x) \\
\text{strain} &= \frac{\text{extension}}{\text{length}} \\
&= \frac{u(x + \mathrm{d}x) - u(x)}{\mathrm{d}x} \approx \frac{\partial u}{\partial x}
\end{aligned}
$$

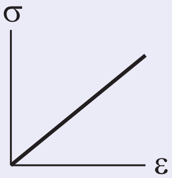
Strain

$$
\varepsilon = \frac{\partial u}{\partial x}
$$

Elasticity

relationship between stress σ and strain ε

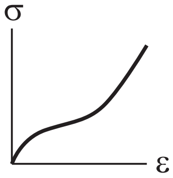
Linear elasticity



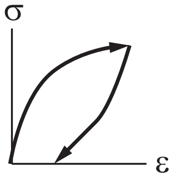
$$\sigma = E\varepsilon$$

E: Young's modulus (elastic modulus)
specific to materials

in reality

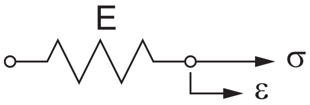


nonlinear

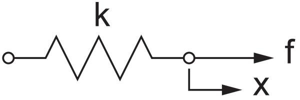


hysteresis

Elasticity

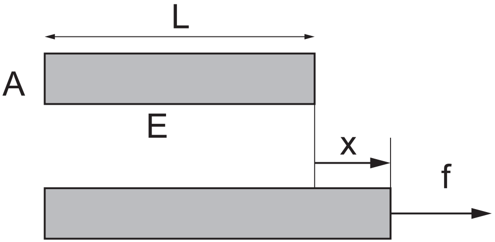


$$\sigma = E\varepsilon$$



$$f = kx$$

extending uniform cylinder

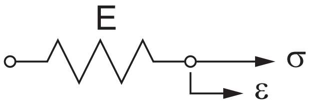


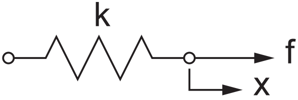
$$f = kx$$

$$k = E\,\frac{A}{L}$$

material geometry

Energy Density



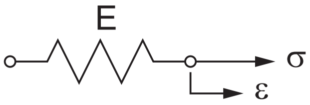


$$U = \frac{1}{2} f x = \frac{1}{2} k x^2$$

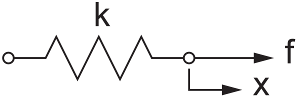
energy

N m

Energy Density



$$\frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$

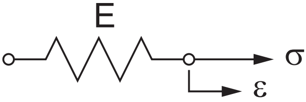


$$U = \frac{1}{2}fx = \frac{1}{2}kx^2$$

energy

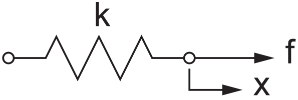
N m

Energy Density



$$\frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$

$$\frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{N\,m}}{\mathrm{m}^3} = \frac{\text{energy}}{\text{volume}}$$

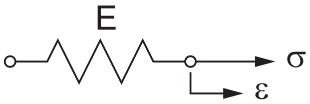


$$U = \frac{1}{2}fx = \frac{1}{2}kx^2$$

energy

N m

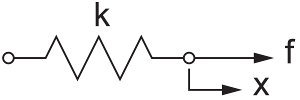
Energy Density



$$\frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$

energy density

$$\frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{N\,m}}{\mathrm{m}^3} = \frac{\text{energy}}{\text{volume}}$$



$$U = \frac{1}{2}fx = \frac{1}{2}kx^2$$

energy

$$\mathrm{N\,m}$$

Strain Potential Energy

energy density of one-dimensional deformation

$$\frac{1}{2}E\varepsilon^2 = \frac{1}{2}E\left(\frac{\partial u}{\partial x}\right)^2$$

volume $\quad A\,\mathrm{d}x$

strain potential energy

$$\begin{aligned} U &= \int_0^L (\text{energy density}) \cdot (\text{volume}) \\ &= \int_0^L \frac{1}{2}E\left(\frac{\partial u}{\partial x}\right)^2 A\,\mathrm{d}x = \int_0^L \frac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \end{aligned}$$

Kinetic Energy

velocity of point $\mathrm{P}(x)$

$$\dot{u} = \frac{\partial u}{\partial t}$$

mass of small region $\mathrm{P}(x)\mathrm{P}(x + \mathrm{d}x)$

$$(\text{density}) \cdot (\text{volume}) = \rho \cdot A \, \mathrm{d}x$$

kinetic energy

$$\begin{aligned} T &= \int_0^L \frac{1}{2} (\text{mass})(\text{velocity})^2 \\ &= \int_0^L \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 \mathrm{d}x \end{aligned}$$

One-dimensional Finite Element Method

energies

strain potential energy

$$U = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x$$

kinetic energy

$$T = \int_0^L \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 \mathrm{d}x$$

How calculate energies in integral forms?

Divide-and-Conquer Approach

divide

$$\int_0^L = \int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \int_{x_3}^{x_4} + \int_{x_4}^{x_5}$$

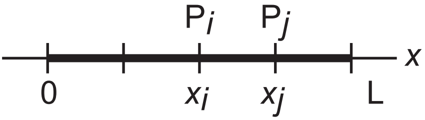
apply piecewise linear approximation

$$\int_{x_i}^{x_j} = \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

synthsize

$$\int_0^L = \frac{1}{2} \begin{bmatrix} u_1 & u_2 & \cdots & u_5 \end{bmatrix} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{bmatrix}$$

Dividing Region

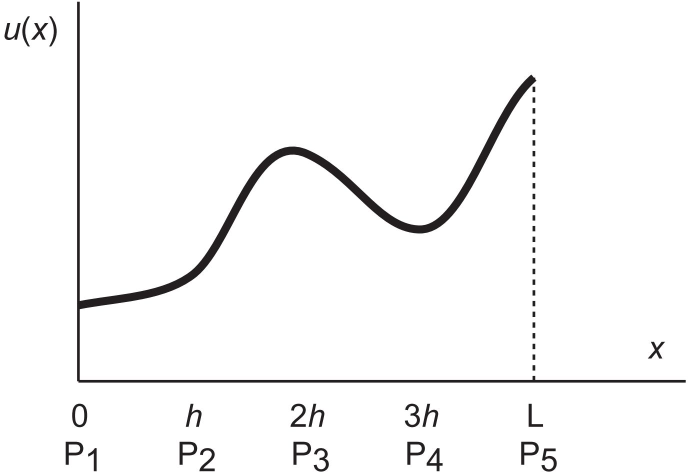


nodal points

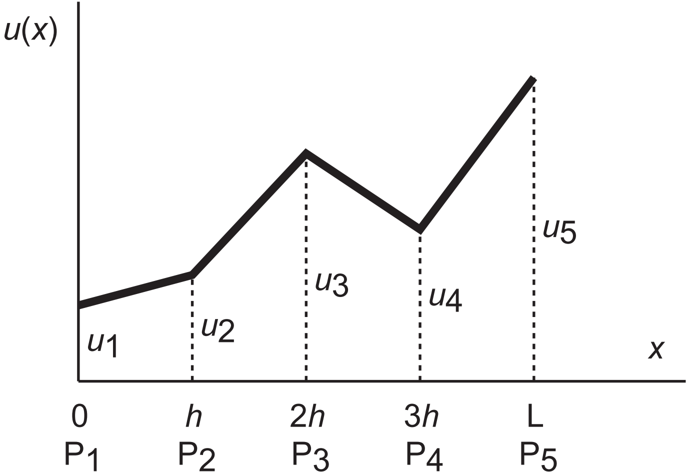
divide $[\,0,\ L\,]$ into four small regions
small region size $h = L/4$

$$x_1 = 0,\ x_2 = h,\ x_3 = 2h,\ x_4 = 3h,\ x_5 = L$$

Piecewise Linear Approximation



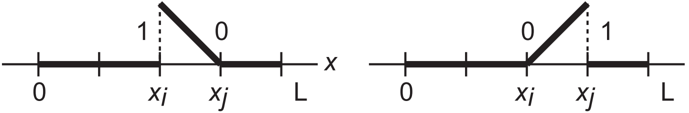
Piecewise Linear Approximation



Piecewise Linear Approximation

function $u(x)$ in small region $[\,x_i,\ x_j\,]$

$$u(x) = u_i\, N_{i,j}(x) + u_j\, N_{j,i}(x)$$



$$N_{i,j}(x) = \frac{x_j - x}{h} = \begin{cases} 1 & (x = x_i) \\ 0 & (x = x_j) \end{cases} \qquad N_{j,i}(x) = \frac{x - x_i}{h} = \begin{cases} 1 & (x = x_j) \\ 0 & (x = x_i) \end{cases}$$

$$u(x_i) = u_i N_{i,j}(x_i) + u_j N_{j,i}(x_i) = u_i \cdot 1 + u_j \cdot 0 = u_i$$
$$u(x_j) = u_i N_{i,j}(x_j) + u_j N_{j,i}(x_j) = u_i \cdot 0 + u_j \cdot 1 = u_j$$

Piecewise Linear Approximation

in small region $[\,x_i,\ x_j\,]$

$$N_{i,j}(x) = \frac{x_j - x}{h}, \qquad N_{j,i}(x) = \frac{x - x_i}{h}$$

$$N'_{i,j}(x) = \frac{-1}{h}, \qquad N'_{j,i}(x) = \frac{1}{h}$$

derivative $\partial u / \partial x$ in small region $[\,x_i,\ x_j\,]$

$$\begin{aligned}\frac{\partial u}{\partial x} &= u_i N'_{i,j}(x) + u_j N'_{j,i}(x) \\ &= u_i \frac{-1}{h} + u_j \frac{1}{h} \\ &= \frac{-u_i + u_j}{h}\end{aligned}$$

Piecewise Linear Approximation

assume Young's modulus E is constant

$$
\begin{aligned}
& \int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \, \mathrm{d}x \\
= & \int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{-u_i + u_j}{h} \right)^2 \, \mathrm{d}x \\
= & \frac{1}{2} \frac{E}{h^2} (-u_i + u_j)^2 \int_{x_i}^{x_j} A \, \mathrm{d}x \\
= & \frac{1}{2} \left[\begin{array}{cc} u_i & u_j \end{array} \right] \frac{E}{h^2} \left[\begin{array}{rr} V_{i,j} & -V_{i,j} \\ -V_{i,j} & V_{i,j} \end{array} \right] \left[\begin{array}{c} u_i \\ u_j \end{array} \right]
\end{aligned}
$$

Piecewise Linear Approximation

note

$$V_{i,j} = \int_{x_i}^{x_j} A \, \mathrm{d}x$$

represents volume in small region $[\, x_i, \, x_j \,]$

assume Young's modulus E and cross-sectional area A are constant

$$\int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x = \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

Synthesizing

nodal displacement vector

$$\boldsymbol{u}_{\mathrm{N}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{bmatrix}$$

describes soft robot deformation

Synthesizing

assume E and A are constant

$$
\begin{aligned}
U = & \frac{1}{2} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
& + \frac{1}{2} \begin{bmatrix} u_2 & u_3 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \\
& + \cdots \\
& + \frac{1}{2} \begin{bmatrix} u_4 & u_5 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_4 \\ u_5 \end{bmatrix}
\end{aligned}
$$

Synthesizing

strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}{}^{\mathrm{T}} K \, \boldsymbol{u}_{\mathrm{N}}$$

stiffness matrix

$$K = \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Synthesizing

strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}{}^{\mathrm{T}} K \, \boldsymbol{u}_{\mathrm{N}}$$

stiffness matrix

$$K = \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 1+1 & -1 & & \\ & -1 & 1+1 & -1 & \\ & & -1 & 1+1 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Synthesizing

strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}{}^{\mathrm{T}} K \, \boldsymbol{u}_{\mathrm{N}}$$

stiffness matrix

$$K = \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 1+1 & -1 & & \\ & -1 & 1+1 & -1 & \\ & & -1 & 1+1 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Piecewise Linear Approximation

in small region $[\, x_i, \, x_j \,]$

$$u = u_i \, N_{i,j} + u_j \, N_{j,i}$$
$$\dot{u} = \dot{u}_i \, N_{i,j} + \dot{u}_j \, N_{j,i}$$

assume density ρ and cross-sectional area A are constant

$$\begin{aligned} \int_{x_i}^{x_j} \frac{1}{2} \rho A \dot{u}^2 \, \mathrm{d}x &= \frac{1}{2} \rho A \begin{bmatrix} \dot{u}_i & \dot{u}_j \end{bmatrix} \begin{bmatrix} h/3 & h/6 \\ h/6 & h/3 \end{bmatrix} \begin{bmatrix} \dot{u}_i \\ \dot{u}_j \end{bmatrix} \\ &= \frac{1}{2} \frac{\rho A h}{6} \begin{bmatrix} \dot{u}_i & \dot{u}_j \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{u}_i \\ \dot{u}_j \end{bmatrix} \end{aligned}$$

Synthesizing

kinetic energy

$$T = \frac{1}{2}\dot{\boldsymbol{u}}_{\mathrm{N}}{}^{\mathrm{T}} M \dot{\boldsymbol{u}}_{\mathrm{N}}$$

inertia matrix

$$M = \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{bmatrix}$$

Dynamic Equation

energies

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}{}^{\mathrm{T}} K \boldsymbol{u}_{\mathrm{N}}$$

$$T = \frac{1}{2} \dot{\boldsymbol{u}}_{\mathrm{N}}{}^{\mathrm{T}} M \dot{\boldsymbol{u}}_{\mathrm{N}}$$

work done by external forces

$$W = \boldsymbol{f}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}}$$

constraints

$$R \stackrel{\triangle}{=} \boldsymbol{a}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}} = 0$$

where $\boldsymbol{f} = [\,0,\ 0,\ 0,\ 0,\ f\,]^{\mathrm{T}}$ and $\boldsymbol{a} = [\,1,\ 0,\ 0,\ 0,\ 0\,]^{\mathrm{T}}$

Dynamic Equation

Lagrangian

$$\mathcal{L} = T - U + W + \lambda_a \boldsymbol{a}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}}$$

λ_a: Lagrange multiplier

Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}_{\mathrm{N}}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}_{\mathrm{N}}} = \mathbf{0}$$

$$-K \boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} + \lambda_a \boldsymbol{a} - M \ddot{\boldsymbol{u}}_{\mathrm{N}} = \mathbf{0}$$

Dynamic Equation

constraint stabilization method

$$\ddot{R} + 2\alpha \dot{R} + \alpha^2 R = 0$$

$$-\boldsymbol{a}^{\mathrm{T}} \ddot{\boldsymbol{u}}_{\mathrm{N}} = 2\alpha \boldsymbol{a}^{\mathrm{T}} \dot{\boldsymbol{u}}_{\mathrm{N}} + \alpha^2 \boldsymbol{a}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}}$$

canonical form of ODE

$$\begin{aligned} \dot{\boldsymbol{u}}_{\mathrm{N}} &= \boldsymbol{v}_{\mathrm{N}} \\ M \dot{\boldsymbol{v}}_{\mathrm{N}} - \lambda_a \boldsymbol{a} &= -K \boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} \\ -\boldsymbol{a}^{\mathrm{T}} \dot{\boldsymbol{v}}_{\mathrm{N}} &= 2\alpha \boldsymbol{a}^{\mathrm{T}} \boldsymbol{v}_{\mathrm{N}} + \alpha^2 \boldsymbol{a}^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}} \end{aligned}$$

Two/Three-dimensional Finite Element Method

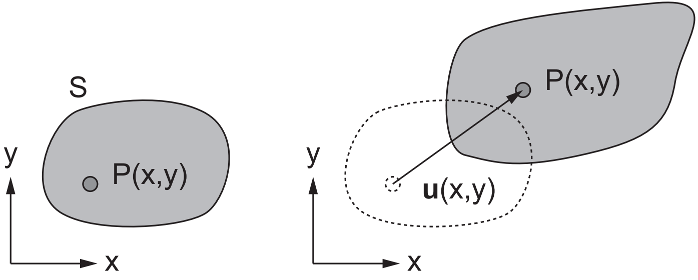
one-dimensional deformation

extensional strain ε
Young's modulus E
strain potential energy density $\quad \dfrac{1}{2}E\varepsilon^2$

two/three-dimensional deformation

extensional & shear strains $\rightarrow$ strain vector $\boldsymbol{\varepsilon}$
Lamé's constants λ, μ $\rightarrow$ elasticity matrix $\lambda I_\lambda + \mu I_\mu$
strain potential energy density $\quad \dfrac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}$

Two-dimensional Deformation

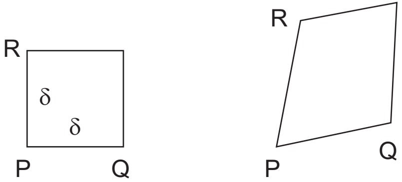


natural state moved and deformed state

displacement vector

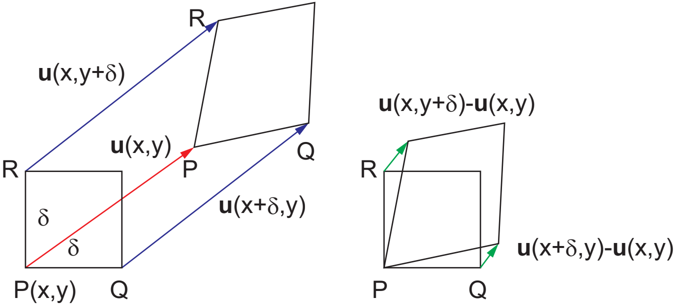
$$\boldsymbol{u}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$$

Two-dimensional Deformation



natural   deformed and rotated

Two-dimensional Deformation



displacements relative displacements

Two-dimensional Deformation

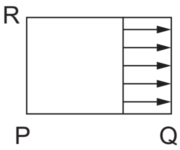
relative displacement at Q

$$\boldsymbol{u}(x+\delta, y) - \boldsymbol{u}(x, y) = \frac{\partial \boldsymbol{u}}{\partial x}\delta = \left[\begin{array}{c} \partial u/\partial x \\ \partial v/\partial x \end{array} \right] \delta$$

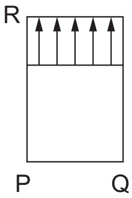
relative displacement at R

$$\boldsymbol{u}(x, y+\delta) - \boldsymbol{u}(x, y) = \frac{\partial \boldsymbol{u}}{\partial y}\delta = \left[\begin{array}{c} \partial u/\partial y \\ \partial v/\partial y \end{array} \right] \delta$$

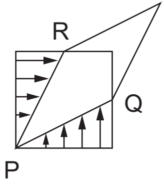
Two-dimensional Deformation



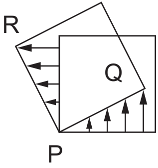
extension along x-axis



extension along y-axis



shear deformation



rotational motion

Two-dimensional Deformation

$$\frac{\partial u}{\partial x} = \text{extension along } x\text{-axis} \quad \frac{\partial u}{\partial y} = \text{shear} - \text{rotation}$$

$$\frac{\partial v}{\partial x} = \text{shear} + \text{rotation} \qquad \frac{\partial v}{\partial y} = \text{extension along } y\text{-axis}$$

$$\Downarrow$$

Cauchy strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Two-dimensional Deformation

strain vector

$$\boldsymbol{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix}$$

Two-dimensional Deformation

Strain potential energy density

linear isotropic elastic material

$$\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_{\lambda} + \mu I_{\mu})\boldsymbol{\varepsilon}$$

where λ and μ are Lamé's constants and

$$I_{\lambda} = \begin{bmatrix} 1 & 1 & \\ 1 & 1 & \\ & & \end{bmatrix}, \quad I_{\mu} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$$

Two-dimensional Deformation

Lamé's constants λ and μ are related to Young's modulus E and Poisson's ratio ν:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$
$$\mu = \frac{E}{2(1+\nu)}$$

Tensile test provides Young's modulus E and Poisson's ratio ν.

Two-dimensional Deformation

Volume element

$$h \, \mathrm{d}S = h \, \mathrm{d}x \, \mathrm{d}y$$

Strain potential energy

$$U = \int_S \frac{1}{2} \boldsymbol{\varepsilon}^{\mathrm{T}} (\lambda I_\lambda + \mu I_\mu) \boldsymbol{\varepsilon} \, h \, \mathrm{d}S$$

Two-dimensional Deformation

Volume element

$$h\,\mathrm{d}S = h\,\mathrm{d}x\,\mathrm{d}y$$

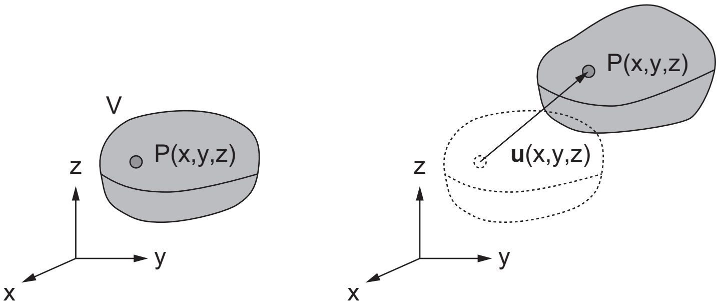
Strain potential energy

$$U = \int_S \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}\,h\,\mathrm{d}S$$

Kinetic energy

$$T = \int_S \frac{1}{2}\rho\,\dot{\boldsymbol{u}}^{\mathrm{T}}\dot{\boldsymbol{u}}\,h\,\mathrm{d}S$$

Three-dimensional Deformation

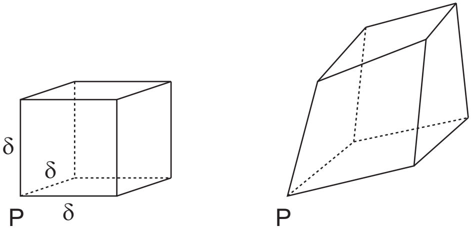


natural state moved and deformed state

displacement vector

$$\boldsymbol{u}(x, y, z) = \begin{bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{bmatrix}$$

Three-dimensional Deformation



natural deformed and rotated

Three-dimensional Deformation

	u	v	w
$\partial/\partial x$	ext. along x	shr $-$ rot in xy	shr $+$ rot in zx
$\partial/\partial y$	shr $+$ rot in xy	ext. along y	shr $-$ rot in yz
$\partial/\partial z$	shr $-$ rot in zx	shr $+$ rot in yz	ext. along z

$$2 \cdot \text{shear in } yz\text{-plane} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$2 \cdot \text{shear in } zx\text{-plane} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$2 \cdot \text{shear in } xy\text{-plane} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Three-dimensional Deformation

Cauchy strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$2\varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Three-dimensional Deformation

strain vector

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{bmatrix}$$

Three-dimensional Deformation

Strain potential energy density

linear isotropic elastic material

$$\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}$$

$$I_\lambda = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array}\right], \quad I_\mu = \left[\begin{array}{ccc|ccc} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right]$$

Three-dimensional Deformation

Volume element

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Strain potential energy

$$U = \int_V \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}\,\mathrm{d}V$$

Three-dimensional Deformation

Volume element

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

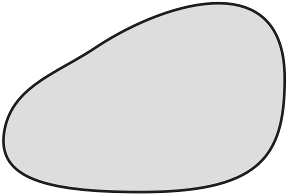
Strain potential energy

$$U = \int_V \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}\,\mathrm{d}V$$

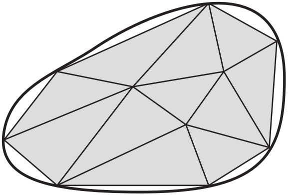
Kinetic energy

$$T = \int_V \frac{1}{2}\rho\,\dot{\boldsymbol{u}}^{\mathrm{T}}\dot{\boldsymbol{u}}\,\mathrm{d}V$$

Two-dimensional FEM

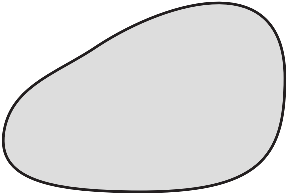


region S

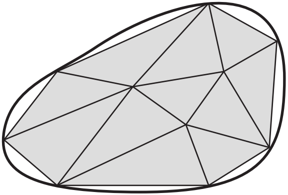


cover by triangles

Two-dimensional FEM



region S



cover by triangles

$$\int_S \mathrm{d}S \approx \sum_{\text{triangles}} \int_{\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k} \mathrm{d}S$$

Two-dimensional FEM

assume density ρ and thickness h are constants
kinetic energy of $\triangle = \triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$

$$T_{i,j,k} = \int_{\triangle} \frac{1}{2} \rho \, \dot{\boldsymbol{u}}^{\mathrm{T}} \dot{\boldsymbol{u}} \, h \, \mathrm{d}S$$

$$= \frac{1}{2} \left[\begin{array}{ccc} \dot{\boldsymbol{u}}_i^{\mathrm{T}} & \dot{\boldsymbol{u}}_j^{\mathrm{T}} & \dot{\boldsymbol{u}}_k^{\mathrm{T}} \end{array} \right] \frac{\rho h \triangle}{12} \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_i \\ \dot{\boldsymbol{u}}_j \\ \dot{\boldsymbol{u}}_k \end{bmatrix}$$

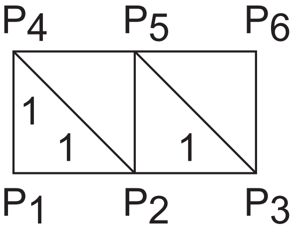
$I_{2\times 2}$: 2×2 identity matrix
(see Finite_Element_Approximation.pdf for details)

Two-dimensional FEM

Partial inertia matrix

$$M_{i,j,k} = \frac{\rho h \triangle}{12} \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix}$$

Example (inertia matrix)



assume $\rho h \triangle / 12$ is constantly equal to 1
partial inertia matrices

$$M_{1,2,4} = M_{2,3,5} = M_{5,4,2} = M_{6,5,3} = \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix}.$$

Example (inertia matrix)

total kinetic energy

$$
T = \frac{1}{2} \dot{\boldsymbol{u}}_{\mathrm{N}}^{\mathrm{T}} \, M \, \dot{\boldsymbol{u}}_{\mathrm{N}}
$$

$$
= \frac{1}{2} \left[\begin{array}{cccc} \dot{\boldsymbol{u}}_1^{\mathrm{T}} & \dot{\boldsymbol{u}}_2^{\mathrm{T}} & \cdots & \dot{\boldsymbol{u}}_6^{\mathrm{T}} \end{array} \right]
\left[\begin{array}{c|c|c|c|c|c}
\quad & \quad & \quad & \quad & \quad & \quad \\ \hline
 & & & & & \\ \hline
 & & & & & \\ \hline
 & & & & & \\ \hline
 & & & & & \\ \hline
 & & & & &
\end{array} \right]
\left[\begin{array}{c} \dot{\boldsymbol{u}}_1 \\ \dot{\boldsymbol{u}}_2 \\ \vdots \\ \dot{\boldsymbol{u}}_6 \end{array} \right]
$$

M: inertia matrix (6×6 block matrix)

Example (inertia matrix)

$$M_{1,2,4} = \left[\begin{array}{c|c|c} (1,1) \text{ block} & (1,2) \text{ block} & (1,4) \text{ block} \\ \hline (2,1) \text{ block} & (2,2) \text{ block} & (2,4) \text{ block} \\ \hline (4,1) \text{ block} & (4,2) \text{ block} & (4,4) \text{ block} \end{array}\right]$$

contribution of $M_{1,2,4}$ to M

$$\left[\begin{array}{c|c|c|c|c|c} 2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & & \\ \hline I_{2\times 2} & 2I_{2\times 2} & & I_{2\times 2} & & \\ \hline & & & & & \\ \hline I_{2\times 2} & I_{2\times 2} & & 2I_{2\times 2} & & \\ \hline & & & & & \\ \hline & & & & & \end{array}\right]$$

Example (inertia matrix)

$$M_{5,4,2} = \left[\begin{array}{c|c|c} (5,5) \text{ block} & (5,4) \text{ block} & (5,2) \text{ block} \\ \hline (4,5) \text{ block} & (4,4) \text{ block} & (4,2) \text{ block} \\ \hline (2,5) \text{ block} & (2,4) \text{ block} & (2,2) \text{ block} \end{array}\right]$$

contribution of $M_{5,4,2}$ to M

$$\left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & 2I_{2\times 2} & & I_{2\times 2} & I_{2\times 2} & \\ \hline & & & & & \\ \hline & I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} & \\ \hline & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2} & \\ \hline & & & & & \\ \hline & & & & & \end{array}\right].$$

Example (inertia matrix)

inertia matrix

$$
\begin{aligned}
M &= M_{1,2,4} \oplus M_{2,3,5} \oplus M_{5,4,2} \oplus M_{6,5,3} \\
&= \begin{bmatrix}
2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & & \\
I_{2\times 2} & 6I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} & 2I_{2\times 2} & \\
& I_{2\times 2} & 4I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} \\
I_{2\times 2} & 2I_{2\times 2} & & 4I_{2\times 2} & I_{2\times 2} & \\
& 2I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} & 6I_{2\times 2} & I_{2\times 2} \\
& & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2}
\end{bmatrix}.
\end{aligned}
$$

Two-dimensional FEM

assume λ, μ and h are constants
strain potential energy stored in $\triangle = \triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$

$$U_{i,j,k} = \int_{\triangle} \frac{1}{2} \boldsymbol{\varepsilon}^{\mathrm{T}} (\lambda I_\lambda + \mu I_\mu) \boldsymbol{\varepsilon} \, h \, \mathrm{d}S$$

$$= \frac{1}{2} \boldsymbol{u}_{i,j,k}^{\mathrm{T}} \left(\lambda J_\lambda^{i,j,k} + \mu J_\mu^{i,j,k} \right) \boldsymbol{u}_{i,j,k}$$

where

$$\boldsymbol{u}_{i,j,k} = \begin{bmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_j \\ \boldsymbol{u}_k \end{bmatrix}$$

(see Finite_Element_Approximation.pdf for details)

Two-dimensional FEM

$$\boldsymbol{a} = \frac{1}{2\triangle} \begin{bmatrix} y_j - y_k \\ y_k - y_i \\ y_i - y_j \end{bmatrix}, \qquad \boldsymbol{b} = \frac{-1}{2\triangle} \begin{bmatrix} x_j - x_k \\ x_k - x_i \\ x_i - x_j \end{bmatrix}$$

$$H_\lambda = \begin{bmatrix} \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} & \boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} & \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} \end{bmatrix} h\triangle$$

$$H_\mu = \begin{bmatrix} 2\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} + \boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} & \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} & 2\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}} + \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} \end{bmatrix} h\triangle$$

$1, 4, 2, 5, 3, 6$ rows and columns of H_λ, H_μ $\rightarrow$
$1, 2, 3, 4, 5, 6$ rows and columns of $J_\lambda^{i,j,k}$, $J_\mu^{i,j,k}$

Example (stiffness matrix)

assume $h = 2$

$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$: $\boldsymbol{a} = [\,-1,\ 1,\ 0\,]^{\mathrm{T}}$ and $\boldsymbol{b} = [\,-1,\ 0,\ 1\,]^{\mathrm{T}}$

$$H_\lambda = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Example (stiffness matrix)

assume $h = 2$

$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$: $\boldsymbol{a} = [\,-1,\ 1,\ 0\,]^{\mathrm{T}}$ and $\boldsymbol{b} = [\,-1,\ 0,\ 1\,]^{\mathrm{T}}$

$$H_{\mu} = \left[\begin{array}{ccc|ccc} 3 & -2 & -1 & 1 & -1 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ \hline 1 & 0 & -1 & 3 & -1 & -2 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \end{array}\right]$$

Example (stiffness matrix)

assume $h = 2$

$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$: $\boldsymbol{a} = [\,-1,\ 1,\ 0\,]^{\mathrm{T}}$ and $\boldsymbol{b} = [\,-1,\ 0,\ 1\,]^{\mathrm{T}}$

$$J_\lambda^{1,2,4} = \left[\begin{array}{rr|rr|rr} 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 & -1 \\ \hline -1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Example (stiffness matrix)

assume $h = 2$

$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$: $\quad \boldsymbol{a} = [\,-1,\ 1,\ 0\,]^{\mathrm{T}}$ and $\boldsymbol{b} = [\,-1,\ 0,\ 1\,]^{\mathrm{T}}$

$$J_{\mu}^{1,2,4} = \left[\begin{array}{rr|rr|rr} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ \hline -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ \hline -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{array}\right]$$

Example (stiffness matrix)

connection matrix

$$J_\lambda = J_\lambda^{1,2,4} \oplus J_\lambda^{2,3,5} \oplus J_\lambda^{5,4,2} \oplus J_\lambda^{6,5,3}$$

$$= \left[\begin{array}{cc|cc|cc|cc|cc|cc}
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
\hline
-1 & -1 & 2 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & & \\
0 & 0 & 1 & 2 & -1 & 0 & 1 & 0 & -1 & -2 & & \\
\hline
& & -1 & -1 & 1 & 0 & & & 0 & 1 & 0 & \\
& & 0 & 0 & 0 & 1 & & & 1 & 0 & -1 & - \\
\hline
0 & 0 & 0 & 1 & & & 1 & 0 & -1 & -1 & & \\
-1 & -1 & 1 & 0 & & & 0 & 1 & 0 & 0 & & \\
\hline
& & 0 & -1 & 0 & 1 & -1 & 0 & 2 & 1 & -1 & - \\
& & -1 & -2 & 1 & 0 & -1 & 0 & 1 & 2 & 0 & \\
\hline
& & & & 0 & -1 & & & -1 & 0 & 1 &
\end{array}\right.$$

Example (stiffness matrix)

connection matrix

$$J_\mu = J_\mu^{1,2,4} \oplus J_\mu^{2,3,5} \oplus J_\mu^{5,4,2} \oplus J_\mu^{6,5,3}$$

$$= \left[\begin{array}{cc|cc|cc|cc|cc|c}
3 & 1 & -2 & -1 & & & -1 & 0 & & & \\
1 & 3 & 0 & -1 & & & -1 & -2 & & & \\ \hline
-2 & 0 & 6 & 1 & -2 & -1 & 0 & 1 & -2 & -1 & \\
-1 & -1 & 1 & 6 & 0 & -1 & 1 & 0 & -1 & -4 & \\ \hline
 & & -2 & 0 & 3 & 0 & & & 0 & 1 & -1 \\
 & & -1 & -1 & 0 & 3 & & & 1 & 0 & 0 \\ \hline
-1 & -1 & 0 & 1 & & & 3 & 0 & -2 & 0 & \\
0 & -2 & 1 & 0 & & & 0 & 3 & -1 & -1 & \\ \hline
 & & -2 & -1 & 0 & 1 & -2 & -1 & 6 & 1 & -2 \\
 & & -1 & -4 & 1 & 0 & 0 & -1 & 1 & 6 & -1 \\ \hline
 & & & & -1 & 0 & & & -2 & -1 & 3
\end{array} \right.$$

Example (stiffness matrix)

stiffness matrix

$$K = \lambda J_\lambda + \mu J_\mu$$

$\lambda,\ \mu$	material-specific
$J_\lambda,\ J_\mu$	geometric

strain potential energy

$$U = \frac{1}{2}\boldsymbol{u}_{\mathrm{N}}^{\mathrm{T}}\, K\, \boldsymbol{u}_{\mathrm{N}}$$

Statics

Variatoinal priciple in statics

$$\text{minimize} \quad I = U - W = \frac{1}{2}\boldsymbol{u}_{\rm N}^{\rm T}\, K\, \boldsymbol{u}_{\rm N} - \boldsymbol{f}\boldsymbol{u}_{\rm N}$$

$$\text{subject to} \quad A^{\rm T}\boldsymbol{u}_{\rm N} = \boldsymbol{b}$$

Intorducing a set of Lagrange multipliers

$$I' = I - \lambda^{\rm T}(A^{\rm T}\boldsymbol{u}_{\rm N} - \boldsymbol{b})$$

$$\Downarrow$$

$$\frac{\partial I'}{\partial \boldsymbol{u}_{\rm N}} = K\,\boldsymbol{u}_{\rm N} - \boldsymbol{f} - A\boldsymbol{\lambda} = \boldsymbol{0}$$

$$\frac{\partial I'}{\partial \boldsymbol{\lambda}} = -(A^{\rm T}\boldsymbol{u}_{\rm N} - \boldsymbol{b}) = \boldsymbol{0}$$

Statics

$$\frac{\partial I'}{\partial \boldsymbol{u}_{\mathrm{N}}} = K\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{f} - A\boldsymbol{\lambda} = \boldsymbol{0}$$

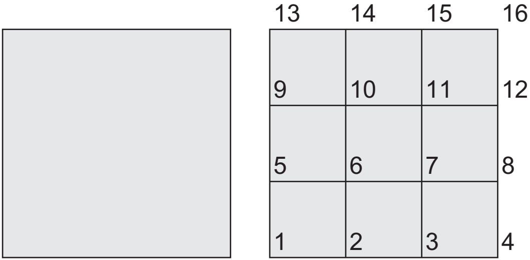
$$\frac{\partial I'}{\partial \boldsymbol{\lambda}} = -(A^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{b}) = \boldsymbol{0}$$

$$\Downarrow$$

Linear equation

$$\begin{bmatrix} K & -A \\ -A^{\mathrm{T}} & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\mathrm{N}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ -\boldsymbol{b} \end{bmatrix}$$

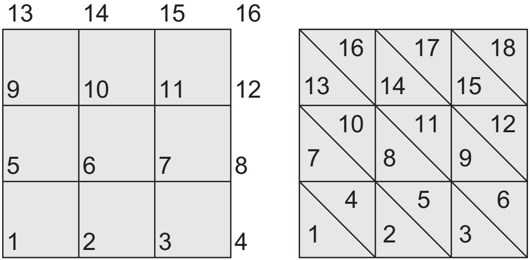
Example (static simulation)



Sample program 'get_started.m'.

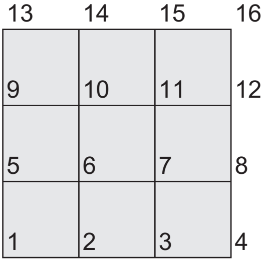
$$\text{points} = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 1 & \cdots & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & \cdots & 3 & 3 \end{bmatrix}$$

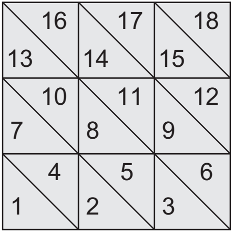
Example (static simulation)



$$\text{triangles} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 3 & 4 & 7 \\ 6 & 5 & 2 \\ & \vdots & \\ 15 & 14 & 11 \\ 16 & 15 & 12 \end{bmatrix}$$

Example (static simulation)






```
npoints = size(points,2);
ntriangles = size(triangles,1);
thickness = 1;
elastic = Body(npoints, points, ntriangles, tria
```

Variable 'elastic' represents the rectangle body.

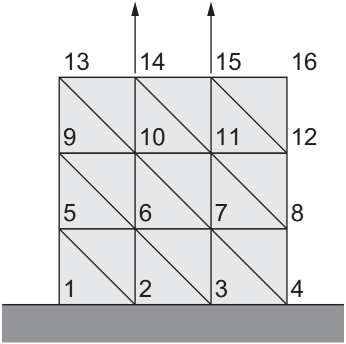
Example (static simulation)

Defining elatic property to calculate stiffness matrix.

```
% E = 0.1 MPa; \nu = 0.48; rho = 1 g/cm^2
Young = 1.0*1e+6; nu = 0.48; density = 1.00;
[ lambda, mu ] = Lame_constants( Young, nu );
elastic = elastic.mechanical_parameters(density

% stiffness matrix
elastic = elastic.calculate_stiffness_matrix;
K = elastic.Stiffness_Matrix;
```

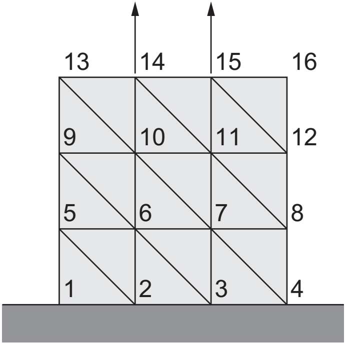
Example (static simulation)



Bottom face is fixed to floor.
Edge $\mathrm{P}_{14}\mathrm{P}_{15}$ is pulled up / pushed down.

$$A^{\mathrm{T}} \boldsymbol{u}_{\mathrm{N}} = \boldsymbol{b}$$

Example (static simulation)



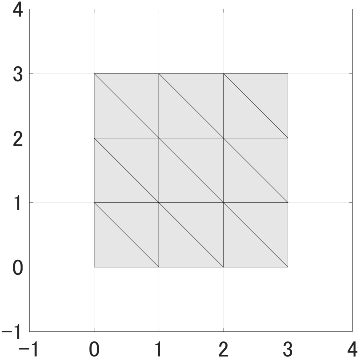

```
% constraints
nconstraints = 12;
A = elastic.constraint_matrix([1, 2, 3, 4, 14,
dy = -0.3;
b = [ 0;0; 0;0; 0;0; 0;0; 0;dy; 0;dy ];
```

Example (static simulation)

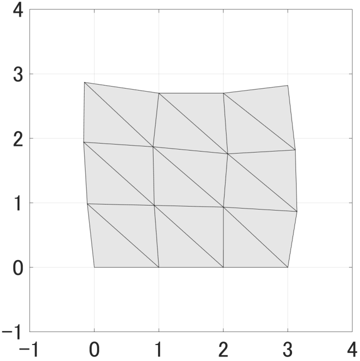
Building and solving linear equation

```
mat = [ K, -A; -A', zeros(nconstraints,nconstra
vec = [ zeros(2*npoints,1); -b ];
sol = mat \ vec;
un = sol(1:2*npoints);
```

Example (static simulation)



natural



deformed

Dynamics

Lagrangian

$$\mathcal{L}(\boldsymbol{u}_{\rm N}, \dot{\boldsymbol{u}}_{\rm N}) = T - U + W + \boldsymbol{\lambda}^{\rm T}\boldsymbol{R}$$
$$= \frac{1}{2}\dot{\boldsymbol{u}}_{\rm N}^{\rm T}\, M\, \dot{\boldsymbol{u}}_{\rm N} - \frac{1}{2}\boldsymbol{u}_{\rm N}^{\rm T}\, K\, \boldsymbol{u}_{\rm N} + \boldsymbol{f}^{\rm T}\boldsymbol{u}_{\rm N} + \boldsymbol{\lambda}^{\rm T}(A^{\rm T}\boldsymbol{u}_{\rm N} - \boldsymbol{b}(t))$$

Partial derivatives

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}_{\rm N}} = -K\boldsymbol{u}_{\rm N} + \boldsymbol{f} + A\boldsymbol{\lambda}, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}_{\rm N}} = M\dot{\boldsymbol{u}}_{\rm N}$$

Lagrange equation of motion

$$-K\boldsymbol{u}_{\rm N} + \boldsymbol{f} + A\boldsymbol{\lambda} - M\ddot{\boldsymbol{u}}_{\rm N} = \boldsymbol{0}$$

Dynamics

Equation for stabilizing constraint $A^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{b}(t) = \boldsymbol{0}$

$$(A^{\mathrm{T}}\ddot{\boldsymbol{u}}_{\mathrm{N}} - \ddot{\boldsymbol{b}}(t)) + 2\alpha(A^{\mathrm{T}}\dot{\boldsymbol{u}}_{\mathrm{N}} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{b}(t)) = \boldsymbol{0}$$

Canonical form

$$\begin{bmatrix} M & -A \\ -A^{\mathrm{T}} & \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}}_{\mathrm{N}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} \\ C(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}) \end{bmatrix}$$

where

$$\boldsymbol{C}(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}) = -\ddot{\boldsymbol{b}}(t) + 2\alpha(A^{\mathrm{T}}\boldsymbol{v}_{\mathrm{N}} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^{\mathrm{T}}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{b}(t))$$

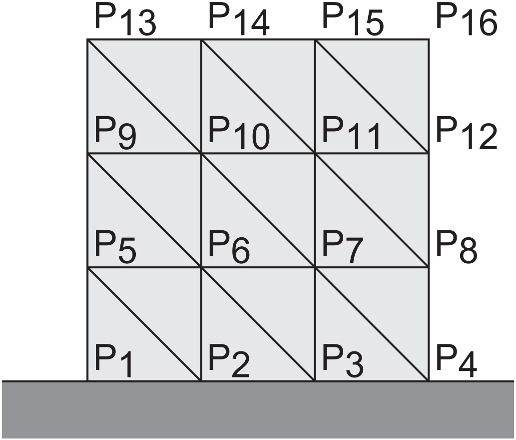
Given $\boldsymbol{u}_{\mathrm{N}}$, $\boldsymbol{v}_{\mathrm{N}}$, we can calcultae time-derivatives $\dot{\boldsymbol{u}}_{\mathrm{N}}$, $\dot{\boldsymbol{v}}_{\mathrm{N}}$.

Example (dynamic simulation)

two-dimensional square soft body of width w

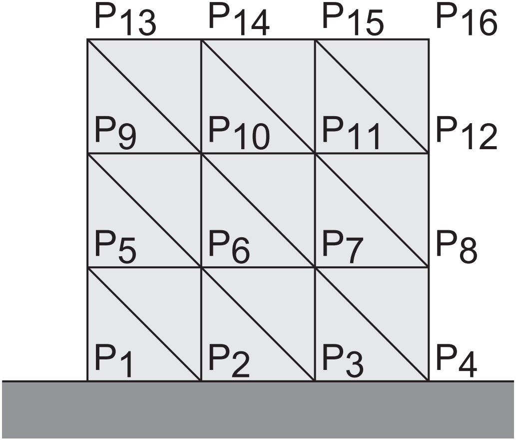
Young's modulus E, viscous modulus c, density ρ

divide square into $3 \times 3 \times 2$ triangles

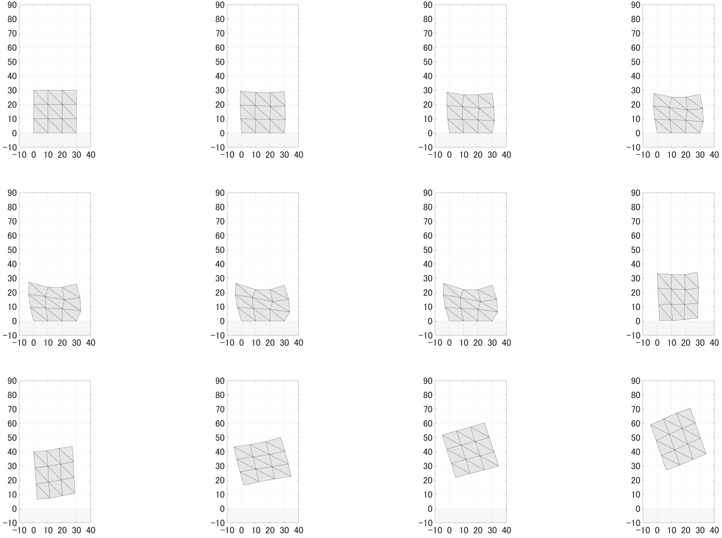


Example (dynamic simulation)

$[\,0,\ t_{push}\,]$	fix the bottom & push $\mathrm{P}_{14}\mathrm{P}_{15}$ downward
$[\,t_{push},\ t_{hold}\,]$	fix the bottom & keep $\mathrm{P}_{14}\mathrm{P}_{15}$
$[\,t_{hold},\ t_{end}\,]$	free (reaction force: penalty method)

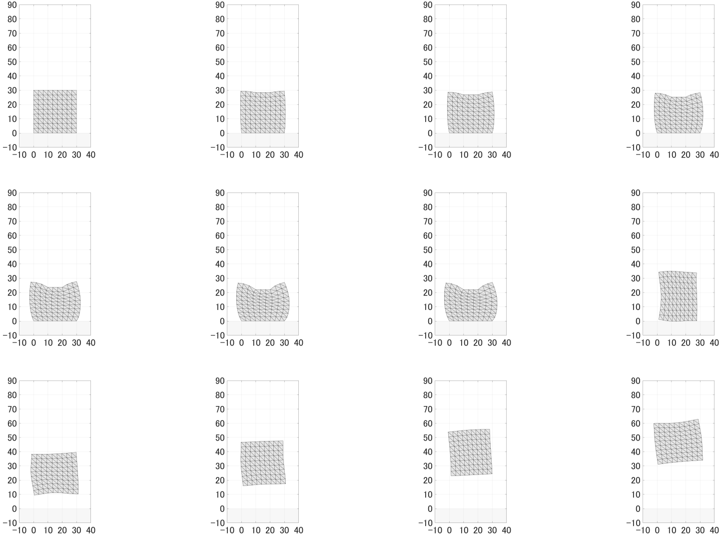


Example (dynamic simulation)



jump simulation movie

Example (dynamic simulation)



jump simulation movie

Example (dynamic simulation)

- motion and deformation can be simulated properly
- results depend on mesh and include artifacts
- finer mesh yields better result but needs more computation time

Summary

energies in integral forms

potential energy

$$U = \int (\text{potential energy density}) \cdot (\text{volume element})$$

kinetic energy

$$T = \int (\text{kinetic energy density}) \cdot (\text{volume element})$$

Summary

integrals

$$\int_{\text{region}} \approx \sum_{\text{small regions}} \int_{\text{small region}}$$

1D	line segments
2D	triangles / rectangles / $\cdots$
3D	tetrahedra / cubes / $\cdots$

Summary

one-dimensional deformation

extensional strain	ε
Young's modulus	E
strain potential energy density	$\dfrac{1}{2}E\varepsilon^2$
kinetic energy density	$\dfrac{1}{2}\rho\dot{\varepsilon}^2$
volume element	$A\,\mathrm{d}x$

Summary

two/three-dimensional deformation

strain vector	$\boldsymbol{\varepsilon}$ (extensional & shear strains)
elasticity matrix	$\lambda \boldsymbol{I}_\lambda + \mu \boldsymbol{I}_\mu$ (Lamé's constants λ, μ)
strain potential energy density	$\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\lambda \boldsymbol{I}_\lambda + \mu \boldsymbol{I}_\mu)\boldsymbol{\varepsilon}$
kinetic energy density	$\frac{1}{2}\rho\dot{\boldsymbol{\varepsilon}}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}}$
volume element	$h\,\mathrm{d}S$ or $\mathrm{d}V$

Summary

strain potential energy

quadratic form with respect to $\boldsymbol{u}_{\mathrm{N}}$

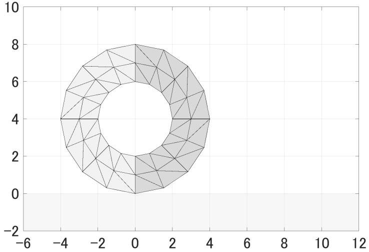
$$U = \frac{1}{2}\boldsymbol{u}_{\mathrm{N}}^{\mathrm{T}}\, K\, \boldsymbol{u}_{\mathrm{N}} \quad (K\text{: stiffness matrix})$$

kinetic energy

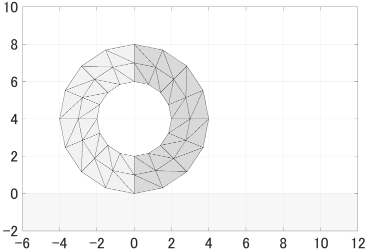
quadratic form with respect to $\dot{\boldsymbol{u}}_{\mathrm{N}}$

$$T = \frac{1}{2}\dot{\boldsymbol{u}}_{\mathrm{N}}^{\mathrm{T}}\, M\, \dot{\boldsymbol{u}}_{\mathrm{N}} \quad (M\text{: inertia matrix})$$

Advances



Cauchy strain (video)



Green strain (video)

Green strain is invariant with respect to rotation whereas Cauchy strain is not

Handouts

Text and sample programs (MATLAB) are available at:

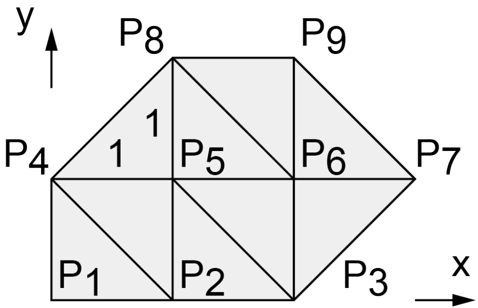
`https://www.hirailab.com/edu/common/soft_robotics/Physics_Soft_Bodies.html`

Report (1/3)

Q1 A soft robot moves inside a smooth rigid tube. The robot body consists of a cylindrical soft tube (length L, outer radius R, inner radius r) and thin rigid plates attached to the both ends of the tube. Young's modulus of the tube materical is given by E. Air pressure P is applied inside the tube through its one end. Assume that the robot extends along its central axis alone and radial deformation is negligible. Let $L = 100$ mm, $R = 10$ mm, $r = 6$ mm, $E = 1.0$ MPa, and $P = 0.10$ MPa, estimate the extentional deformation of the robot.

Report (2/3)

Q2 Show inertia matrix M and connection matrices J_λ, J_μ of the two-dimensional body below. Length of orthogonal sides of all isosceles right triangles is 1. Thickness of the two-dimensional body is $h = 2$ and its density is $\rho = 12$.



Report (3/3)

Submit your report in PDF format through manaba+R.
Other format files are not accepted.
due :00:10 am, November 8 (Friday).
Either English or 日本語 is accepted.