Analytical Mechanics Report

due date: Janurry 29, 2007
should be submitted to Hirai's room (East 4F)

1. Let us investigate the one-dimensional extension of an irregular-shaped beam. Let s be the distance from one end point in the natural state. Let $\mathrm{P}(s)$ be point on the beam at distance s, as illustrated in Figure 1. Let $u(s)$ be the displacement of point $\mathrm{P}(s)$. Let us divide region $[0, L]$ into n regions with interval $h = L/n$. Let $u_i(t)$ be the displacement of the i-th nodal point $\mathrm{P}(s_i)$, where $s_i = hi$.

Let $A(s)$ and $E(s)$ be be cross-sectional area and Young's modulus at $\mathrm{P}(s)$. Let us approximate function $E(s)A(s)$ linearly in region $[s_i, s_j]$ as

$$E(s)A(s) = EA_i N_{i,j}(s) + EA_j N_{i,j}(s),$$

where $EA_i = E(s_i)A(s_i)$ and $EA_j = E(s_j)A(s_j)$. Show that potential energy U can be approximated by

$$U = \frac{1}{2}\boldsymbol{u}_{\mathrm{N}}{}^{\mathrm{T}} K \boldsymbol{u}_{\mathrm{N}}$$

where

$$\boldsymbol{u}_{\mathrm{N}} = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_n \end{bmatrix}^{\mathrm{T}},$$

$$K = \frac{1}{h}\begin{bmatrix} e_{0,1} & -e_{0,1} & & & & \\ -e_{0,1} & e_{0,1}+e_{1,2} & -e_{1,2} & & & \\ & -e_{1,2} & e_{1,2}+e_{2,3} & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & e_{n-2,n-1}+e_{n-1,n} & -e_{n-1,n} \\ & & & & -e_{n-1,n} & e_{n-1,n} \end{bmatrix},$$

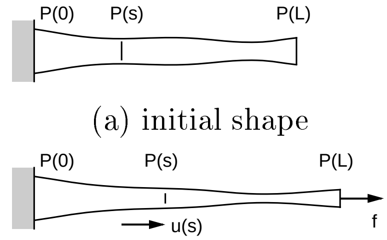
and $e_{i,j} = (EA_i + EA_j)/2$.

2. Let us describe rotation matrix R by four parameters q_0, q_1, q_2, and q_3 as follows:

$$R = \begin{bmatrix} 2(q_0^2+q_1^2)-1 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & 2(q_0^2+q_2^2)-1 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & 2(q_0^2+q_3^2)-1 \end{bmatrix},$$

where the four parameters must satisfy the following geometric constraint:

$$Q \triangleq q_0^2 + q_1^2 + q_2^2 + q_3^2 - 1 = 0.$$



(a) initial shape

(b) deformed shape

Figure 1: Extension of irregular-shaped beam

This description is referred to as *quaternions*. Elements in the rotation matrix are given by quadratic expressions without trigonometric functions, resulting that the rotation matrix is free from singularity. Thus, this method is widely used in control of satellite and computer graphics.
(1) Show that the above matrix R is orthogonal.
(2) Describe angular velocities ω_ξ, ω_η, and ω_ζ by parameters q_0, q_1, q_2, and q_3 and their time-derivatives $\dot{q}_0$, $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$.
(3) Describe time-derivatives $\dot{q}_0$, $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$ by angular velocities ω_ξ, ω_η, and ω_ζ and parameters q_0, q_1, q_2, and q_3.
(5) Applying the constraint stabilization method (CSM) for non-holonomic constraints, show equations to compute parameters q_0, q_1, q_2, and q_3 from angular velocities ω_ξ, ω_η, and ω_ζ.
(5) Show that rotation around a unit vector $\boldsymbol{u} = [u_x, u_y, u_z]^{\mathrm{T}}$ by angle α is formulated by rotation matrix given by

$$S(\boldsymbol{u}, \alpha) =$$
$$\begin{bmatrix} C_\alpha + (1 - C_\alpha)u_x^2 & (1 - C_\alpha)u_x u_y - S_\alpha u_z & (1 - C_\alpha)u_x u_z + S_\alpha u_y \\ (1 - C_\alpha)u_y u_x + S_\alpha u_z & C_\alpha + (1 - C_\alpha)u_y^2 & (1 - C_\alpha)u_y u_z - S_\alpha u_x \\ (1 - C_\alpha)u_z u_x - S_\alpha u_y & (1 - C_\alpha)u_y u_z + S_\alpha u_x & C_\alpha + (1 - C_\alpha)u_z^2 \end{bmatrix},$$

where $C_\alpha = \cos\alpha$ and $S_\alpha = \sin\alpha$. In addition, letting $q_0 = \cos(\alpha/2)$, $q_1 = u_x \sin(\alpha/2)$, $q_2 = u_y \sin(\alpha/2)$, and $q_3 = u_z \sin(\alpha/2)$, show that the above equation coincides to the rotation matrix described by quaternions.